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Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces

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Abstract. We show that if $\mathbf{G}$ is a semisimple algebraic group defined o
arithmetic lattice in $G := \mathbf{G}_{\mathbf{R}}$ with respect to the $\mathbf{Q}$-structure, then ther
subset C of G/Γ such that, for any unipotent one-parameter subgroup
$g \in G$, the time spent in C by the $\{u_t\}$-trajectory of $g\Gamma$, during the time
asymptotic to T, unless $\{g^{-1}u_t g\}$ is contained in a $\mathbf{Q}$-parabolic subg
quantitative versions of this are also proved. The results strengthen sim
$SL(n, \mathbf{Z})$, $n \geqslant 2$, proved earlier in [5] and also enable verification of a t
introduced in [7] for lattices in $SL(3, \mathbf{R})$, which was used in our proof
conjecture for a class of unipotent flows, in [8].

Keywords. Homogeneous spaces; unipotent flows; trajectories.

Margulis [10] showed that if $\{u_t\}$ is a unipotent one-paramete
$G = SL(n, \mathbf{R})$ and $g \in G$ then there exists a compact subset C of $SL(n, \mathbf{R}$
that the set $\{t \geqslant 0 \mid u_t g(SL(n, \mathbf{Z})) \in C\}$ is unbounded. The result played
role in one of the proofs of the arithmeticity theorem for lattices (cf. [1
[5], motivated by certain problems on orbits and invariant measures o
flows, the first named author improved the result. In [3] it was con
$G = SL(n, \mathbf{R})$ and $\Gamma = SL(n, \mathbf{Z})$, given $\varepsilon > 0$ there exists a compact su
such that for any unipotent one-parameter subgroup $\{u_t\}$ and any g

$$\ell(\{t \in [0, T] \mid u_t g\Gamma \in C\}) \geqslant (1 - \varepsilon)\, T$$

for all large T (ℓ being the Lebesgue measure on $\mathbf{R}$) or there exists a

This set of ideas was again involved in [7] where we proved that if
of $SL(3,\mathbf{R})$ of all elements leaving invariant a non-degenerate in
form in 3 variables then every H-orbit on $SL(3,\mathbf{R})/SL(3,\mathbf{Z})$ is eithe
and used this result to conclude in particular that the set of value
B is a nondegenerate indefinite quadratic form in $n \geqslant 3$ variables a
of primitive elements in $\mathbf{Z}^n$, is dense in $\mathbf{R}$ whenever B is not a mul
quadratic form; the latter result strengthened the theorem of the seco
proving a conjecture of Oppenheim (cf. [12]). The proof used a so
result from [5] yielding a version of the above mentioned re
quantitative condition in the second alternative. It was noted tha
theorem about H-orbits on $SL(3,\mathbf{R})/SL(3,\mathbf{Z})$ would go through fo
the place of $SL(3,\mathbf{Z})$, if it satisfied a condition which was called Co
Remark 1·8). While the result from [5] alluded to above is sufficien
$SL(3,\mathbf{Z})$ satisfies Condition $(*)$, it does not yield such a result fo
$SL(3,\mathbf{R})$. This is because, though any lattice in $SL(3,\mathbf{R})$ is arithmet
argument used earlier is not adequate, since the subgroup L in the
there is in general not insured to be contained in a parabolic sub
for an intrinsic approach to proving analogues of the results in [
behaviour of the trajectories, in the case of a general arithmeti
purpose of this paper to carry this out. In particular we shall ve
$(*)$ for any lattice Γ in $SL(3,\mathbf{R})$. It may be mentioned that the con
in our more recent paper [8] where we describe the orbit-closu
unipotent one-parameter subgroup on $SL(3,\mathbf{R})/\Gamma$, Γ any lattice, ver
of Raghunathan for the case. We now introduce some notation an

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbf{Q}$ and let
of $\mathbf{R}$-elements of $\mathbf{G}$. Let r be the $\mathbf{Q}$-rank of $\mathbf{G}$. We suppose that
maximal $\mathbf{Q}$-split torus in $\mathbf{G}$. We fix an order on the system of $\mathbf{Q}$
and denote by $\{\alpha_1, \ldots, \alpha_r\}$ the corresponding system of simple $\mathbf{Q}$-
$i = 1, \ldots, r$ let $\mathbf{P}_i$ be the standard maximal $\mathbf{Q}$-parabolic subgroup
the set of simple roots other than α_i. For each i the root α_i, which
$\mathbf{S}$, extends uniquely to a character on $\mathbf{P}_i$; the extension will also
For $1 \leqslant i \leqslant r$, let $\mathbf{U}_i$ be the unipotent radical of $\mathbf{P}_i$ and $\mathscr{U}_i$ be the
Then there is a positive integer m_i, such that for any $x \in \mathbf{P}_i$ det
equivalently m_i can be defined to be the sum $\Sigma n_\lambda \lambda_i$ taken over a
where for each λ, n_λ is the dimension of the root subspace corresp
is the coefficient of α_i in the expansion of λ in terms of $\alpha_1, \ldots, \alpha_r$.

Let S and P_i, $i = 1, \ldots, r$, denote the subgroups of G consisting

system of simple **Q**-roots defined by $\alpha_i(\text{diag}\,(a_1,\ldots,a_n)) = a_{i+1}/a$
the standard basis of $\mathbf{R}^n$ and for $i = 1,\ldots,n-1$ let Δ_i be the (
generated by $\{e_1,\ldots,e_i\}$. Let K be the subgroup of $G = SL($
orthogonal matrices. Then for $1 \leqslant i \leqslant n-1$ and $g \in G$, $d_i(g)$ as ab
be the same as $d^2(g\Delta_i)$ with d as in [5]; since both the functions
is enough to check their equality for g in P_i.

We now state the main technical result of the paper. It gives a
in terms of d_i, $i = 1,\ldots,r$, for the Lebesgue measure of the set of
an interval, to a certain compact set to be large. The Lebesgue m
denoted by ℓ.

Theorem 1. *Let the notation be as above. Further let* $\Gamma \subset \mathbf{G}$ *be a*
in G *with respect to the* Q*-structure on* $\mathbf{G}$*. Then there exists a fi*
such that the following holds: for any $\varepsilon > 0$ *and* $\theta > 0$ *there exists*
of G/Γ *such that for any unipotent one-parameter subgroup* $\{u_t\}$
any $T \geqslant 0$ *either*

$$\ell(\{t \in [T, \sigma T] \mid u_t g\Gamma \in C\}) \geqslant (1-\varepsilon)(\sigma - 1)\,T$$

for all $\sigma > 1$ *such that* $(1-\sigma^{-1})^r > 1-\varepsilon$*, or there exist* $i \in \{1,\ldots,r\}$*,*
that

$$d_i(u_t g\gamma f) < \theta \quad \forall t \in [0, T].$$

Remarks 1. The set F is so chosen to be the set of inverses of a se
for the standard fundamental domain for Γ in G (cf. [1], Theorem
to the triple (K, P, S) with K and S as above and P the standard
subgroup corresponding to the system $\{\alpha_1,\ldots,\alpha_r\}$ of simple **Q**
details about the set; it can be chosen to be any $F \subset \mathbf{G}_{\mathbf{Q}}$ such tha
notation as in (1.2).
2. In the case of $G = SL(n, \mathbf{R})$ and $\Gamma = SL(n, \mathbf{Z}), n \geqslant 2$, the secon
conclusion of the theorem can be seen to be equivalent to the c
exists a nonzero subgroup Δ of $\mathbf{Z}^n$ such that $d^2(u_t g\Delta) < \theta$ for all

The proof of Theorem 1 will be completed in §3. In §4 we sh
consequences of Theorem 1, which we now describe. For this
$i = 1,\ldots,r$ let $Q_i = \{x \in P_i \mid \alpha_i(x) = 1\}$.

Theorem 2. *Let the notation be as before. Also let* F *be a finite sub*
the contention of Theorem 1 *holds. Then for any* $\varepsilon > 0$ *and* $\theta > 0$ *th*

applies to any unipotent subgroup, rather than a subgroup of P_0, after appropriate modifications; the compact set for a conjugate would be different, however.

A subgroup V of P_0 is said to be in *general position* (relative to S and the order on the roots) if for any $i\in\{1,\ldots,r\}$ and $x\in G$, $xVx^{-1}\subset P_i$ if and only if $x\in P_i$.

Theorem 3. *Let the notation be as above. Then there exists a compact subset C of G/Γ such that the following holds: If V is a connected Lie subgroup of P_0 which consists of unipotent elements and is in general position and $\{x_k\}$ is a sequence in P_0 such that $d_i(x_k)\to\infty$ for all $i=1,\ldots,r$, then for any $g\in G$, $C\cap Vx_kg\Gamma/\Gamma$ is nonempty for all large k. In particular, if R is the subgroup generated by V and $\{x_k|k=1,2\ldots\}$ then every R-orbit on G/Γ intersects C.*

As stated before, one of our aims here is also to verify a technical condition on lattices in $SL(3,\mathbf{R})$ introduced in [7]; namely Condition $(*)$ recalled below. In [7] it was noted that the arguments in the proof of Theorem 2 there went through for any lattice satisfying Condition $(*)$ in the place of $SL(3,\mathbf{Z})$; for the lattice $SL(3,\mathbf{Z})$ the condition was verified using the results in [5]. We had mentioned that the condition in fact holds for all lattices but did not go into the proof, as our primary interest in that paper lay in the lattice $SL(3,\mathbf{Z})$. The condition is also used in the more recent paper [8] where we obtain a full description of orbit closures of generic unipotent one-parameter subgroups on $SL(3,\mathbf{R})/\Gamma$, Γ any lattice in $SL(3,\mathbf{R})$, verifying a conjecture of Raghunathan for the case.

For each $t\in\mathbf{R}$ let

$$v_1(t)=\begin{pmatrix}1 & t & \frac{1}{2}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}$$

and let V_1 be the subgroup $\{v_1(t)|t\in\mathbf{R}\}$. A lattice Γ in $SL(3,\mathbf{R})$ is said to satisfy Condition $(*)$ if there exists a compact subset C of G/Γ such that for any $g\in G$ the following conditions hold:

a) the sets $\{t\geqslant 0|v_1(t)g\Gamma\in C\}$ and $\{t\leqslant 0|v_1(t)g\Gamma\in C\}$ are both unbounded unless there exists a proper parabolic subgroup P of $SL(3,\mathbf{R})$ such that if L is the closed subgroup generated by all unipotent elements in P then $g^{-1}V_1g\subset L$, $L\Gamma$ is closed and $L\cap\Gamma$ is a lattice in L and

b) if $\{f(t)\}_{t\geqslant 0}$ is a curve in $N(V_1)$ (the normalizer of V_1) such that $|\det f(t)|W|\to\infty$ as $t\to\infty$ for every proper nonzero $N(V_1)$-invariant subspace W of $\mathbf{R}^3$ then $C\cap V_1f(t)g\Gamma/\Gamma$ is nonempty for all large t.

Theorem 4. *Any lattice in $SL(3,\mathbf{R})$ satisfies Condition $(*)$.*

Now let I be any (possibly empty) subset of $\{1,\ldots,r\}$. We define

$$\mathbf{P}_I = \cap_{i\in I}\mathbf{P}_i, \quad \mathbf{Q}_I = \cap_{i\in I}\mathbf{Q}_i \text{ and } \mathbf{S}_I = \Pi_{i\in I}\mathbf{S}_i.$$

Then $\mathbf{P}_i$ is the standard parabolic $\mathbf{Q}$-subgroup corresponding to the subset of $\{\alpha_1,\ldots,\alpha_r\}$ complementary to I (in particular $\mathbf{P}_\phi = \mathbf{G}$), $\mathbf{Q}_I$ is a normal algebraic $\mathbf{Q}$-subgroup of $\mathbf{P}_I$, $\mathbf{S}_I$ is a $\mathbf{Q}$-split torus and $\mathbf{P}_I = \mathbf{S}_I\mathbf{Q}_I$. Let $\mathbf{U}_I$ be the unipotent radical of $\mathbf{P}_I$ (and also $\mathbf{Q}_I$) and let $\mathbf{H}_I$ be the centraliser of $\mathbf{S}_I$ in $\mathbf{Q}_I$. Then $\mathbf{Q}_I = \mathbf{H}_I\mathbf{U}_I$ (semidirect product). We also note that $\mathbf{H}_I$ and $\mathbf{U}_I$ are defined over $\mathbf{Q}$. We denote by P_I, Q_I, S_I, H_I and U_I the subgroups of G consisting of $\mathbf{R}$-elements of $\mathbf{P}_I$, $\mathbf{Q}_I$, $\mathbf{S}_I$, $\mathbf{H}_I$ and $\mathbf{U}_I$ respectively.

Since $\mathbf{H}_I$ is defined over $\mathbf{Q}$, $\Gamma\cap H_I$ is an arithmetic subgroup of H_I. It is easy to see that there is no nontrivial character on $\mathbf{H}_I$ defined over $\mathbf{Q}$. Therefore $\Gamma\cap H_I$ is a lattice in H_I. If $I = \{1,\ldots,r\}$, H_I is of $\mathbf{Q}$-rank 0 and hence $\Gamma\cap H_I$ is a uniform lattice in H_I; that is, $H_I/\Gamma\cap H_I$ is compact. Since $\mathbf{U}_I$ is a unipotent algebraic subgroup defined over $\mathbf{Q}$, $U_I/\Gamma\cap U_I$ is also compact. Thus in the case $I = \{1,\ldots,r\}$, $Q_I/\Gamma\cap Q_I$ is compact.

Now let I be any (possibly empty) proper subset of $\{1,\ldots,r\}$ and let $J = \{1,\ldots,r\} - I$. We note that $\mathbf{S}_J$ is a maximal $\mathbf{Q}$-split torus in $\mathbf{H}_I$, $\mathbf{P}_J\cap\mathbf{H}_I$ is a minimal $\mathbf{Q}$-parabolic subgroup of $\mathbf{H}_I$ and $\mathbf{U}_J\cap\mathbf{H}_I$ is the unipotent radical of $\mathbf{P}_J\cap\mathbf{H}_I$. We note next that since, by choice, the Cartan involution associated to K leaves S invariant, it also follows that it leaves H_I invariant. This implies that $K\cap H_I$ is a maximal compact subgroup of H_I. Corresponding to the triple $(K\cap H_I, P_J\cap H_I, S_J)$ there exists a $t_I > 0$, a compact subset C_I of $U_J\cap H_I$ and a finite subset E_I of $\mathbf{G}_{\mathbf{Q}}\cap H_I$ such that

$$H_I = (K\cap H_I)\Omega(t_I)C_I E_I(\Gamma\cap H_I),$$

where

$$\Omega(t_I) = \{s\in S_J \mid 0 < \alpha_j(s) \leqslant t_I \quad \forall j\in J\}$$

(cf. [1] Theorem 13.1). Since $\mathbf{U}_I$ is a unipotent algebraic $\mathbf{Q}$-group, the arithmetic subgroup $\Gamma\cap U_I$ is a uniform lattice in U_I (that is, $U_I/\Gamma\cap U_I$ is compact) and hence there exists a compact subset D_I of U_I such that $U_I = D_I(\Gamma\cap U_I)$. Then we have

$$\begin{aligned} Q_I = H_I U_I &= (K\cap H_I)\Omega(t_I)C_I E_I(\Gamma\cap H_I)U_I \\ &= (K\cap H_I)\Omega(t_I)C_I U_I E_I(\Gamma\cap H_I) \\ &= (K\cap H_I)\Omega(t_I)C_I D_I(\Gamma\cap U_I)E_I(\Gamma\cap H_I). \end{aligned}$$

It is easy to see that since $E_I \subset \mathbf{G}_{\mathbf{Q}}\cap H_I$ there exists a finite subset F_I of $\mathbf{G}_{\mathbf{Q}}\cap Q_I$ such that

$$(\Gamma\cap U_I)E_I(\Gamma\cap H_I) \subset F_I(\Gamma\cap Q_I).$$

Hence we have

$$Q_I = (K\cap H_I)\Omega(t_I)\mathfrak{U}_I F_I(\Gamma\cap Q_I) \qquad (1.1)$$

We shall use the facts mentioned above and the notation to deduce compactness of certain sets which we now introduce.

A p-tuple $((i_1, \lambda_1), \ldots, (i_p, \lambda_p))$, where $p \geqslant 1$, $i_1, \ldots, i_p \in \{1, \ldots, r\}$ and $\lambda_1, \ldots, \lambda_p \in \mathbf{G}_{\mathbf{Q}}$ is called an *admissible* sequence of length p if $i_1, \ldots, i_p$ are distinct and $\lambda_{j-1}^{-1} \lambda_j \in \Lambda(\{i_1, \ldots, i_{j-1}\})$ for all $j = 1, \ldots, p$, λ_0 being taken to be the identity element. The empty sequence is called an admissible sequence of length 0. If ξ and η are two admissible sequences of lengths p and q respectively and $p \leqslant q$ then η is said to extend ξ if the first p terms of η coincide with the corresponding terms of ξ; any admissible sequence extends the empty sequence.

For any admissible sequence ξ of length $p \geqslant 0$ we denote by $\mathscr{C}(\xi)$ the set of all pairs (i, λ), where $1 \leqslant i \leqslant r$ and $\lambda \in \mathbf{G}_{\mathbf{Q}}$, for which there exists an admissible sequence η of length $p+1$ extending ξ and containing (i, λ) as a (necessarily the last) term; note that if $p = 0$, namely if ξ is the empty sequence, $\mathscr{C}(\xi)$ consists of all (i, λ) where $1 \leqslant i \leqslant r$ and $\lambda \in \Lambda(\phi)$.

For any admissible sequence ξ of length $p \geqslant 0$ we define the *support* of ξ, to be the empty set if $p = 0$ and the set $\{(i_1, \lambda_1), \ldots, (i_p, \lambda_p)\}$ if $\xi = ((i_1, \lambda_1), \ldots, (i_p, \lambda_p))$; the support of ξ will be denoted by supp ξ.

The main result on compact subsets of G/Γ needed in the sequel is the following:

PROPOSITION 1.3

Let ξ be an admissible sequence of length $p \geqslant 0$. Let α, a and b be positive real numbers and let

$$W = \{g \in G \mid d_i(g\lambda) \geqslant \alpha \text{ for all } (i, \lambda) \in \mathscr{C}(\xi) \text{ and}$$
$$a \leqslant d_i(g\lambda) \leqslant b \text{ for all } (i, \lambda) \in \text{supp } \xi\}.$$

Then $W\Gamma/\Gamma$ is contained in a compact subset of G/Γ.

For proving the proposition we need the following Lemmas.

Lemma 1.4. *Let $i \in \{1, \ldots, r\}$ and let C be a compact subset of G. Then there exists a $c > 0$ such that $d_i(xg) \geqslant c d_i(g) \forall x \in C$ and $g \in G$.*

Proof. Recall that $G = KP_i$. Since CK is a compact subset of G there exists a compact subset D of P_i such that $CK \subset KD$. Since D is compact and α_i is continuous, there exists a $c > 0$ such that $|\alpha_i(y)|^{m_i} \geqslant c$ for all $y \in D$. Now let $x \in C$ and $g \in G$ be given. Then there exist $k \in K$ and $h \in P_i$ such that $g = kh$. Further, by the choice of D, there exist $k' \in K$ and $y \in D$ such that $xk = k'y$. Then $xg = xkh = k'yh$ and hence

$$d_i(xg) = d_i(k'yh) = |\alpha_i(yh)|^{m_i} = |\alpha_i(y)|^{m_i} |\alpha_i(h)|^{m_i}$$
$$\geqslant c |\alpha_i(h)|^{m_i} = c d_i(g)$$

which proves the Lemma.

Lemma 1.5. *Let I be a subset of $\{1, \ldots, r\}$ and let $j \in \{1, \ldots, r\} - I$. Let $0 < a \leqslant b$ be given. Then there exists a compact subset K_0 of Q_I such that if $g \in Q_I$ and $d_j(g) \in [a, b]$*

Proof. Since $K\cap H_I$ is a maximal compact subgroup of H_I and $P_j\cap H_I$ is a parabolic subgroup of H_I we have $H_I=(K\cap H_I)(P_j\cap H_I)$. Hence $Q_I=H_I\cdot U_I=(K\cap H_I)\cdot(P_j\cap H_I)U_I=(K\cap H_I)(P_j\cap Q_I)$. It is also easy to see, by comparing the root subgroups on either side, that $P_j\cap Q_I=S_jQ_{I\cup\{j\}}$. Thus $Q_I=(K\cap H_I)S_jQ_{I\cup\{j\}}$. Further, for $g\in Q_I$ expressed as $g=ksh$ with $k\in K\cap H_I$, $s\in S_j$ and $h\in Q_{I\cup\{j\}}$ we have $d_j(g)=|\alpha_j(s)|^{m_j}$. This shows that if $g\in Q_I$ and $d_j(g)\in[a,b]$ then $g\in K_0Q_{I\cup\{j\}}$, where $K_0=(K\cap H_I)\cdot\{s\in S_j\,\|\,\alpha_j(s)|^{m_j}\in[a,b]\}$. Since K_0 is a compact subset of Q_I, this proves the Lemma.

Proof of Proposition 1.3. First let $p=0$, namely let ξ be the empty sequence. Then we see that $W=\{g\in G\,|\,d_i(g\lambda)\geqslant\alpha$ for all $i=1,\ldots,r$ and $\lambda\in\Lambda(\phi)\}$. Let $g\in W$. By the particular case of (1.1) with $I=\phi$, g (in fact, any element of G) can be expressed as $kw\psi f$ where $k\in K$, $w\in\Omega(t_\phi)$, $\psi\in\Psi_\phi$ and $f\in F_\phi\Gamma=\Lambda(\phi)^{-1}$. Consider such a decomposition and let $\lambda=f^{-1}\in\Lambda(\phi)$. Then we see that for any $i=1,\ldots,r$

$$|\alpha_i(w)|^{m_i}=d_i(kw\psi)=d_i(g\lambda)\geqslant\alpha.$$

This shows that

$$W\subseteq K\Omega_0\Psi_\phi F_\phi\Gamma \tag{1.6}$$

where $\Omega_0=\{w\in\Omega(t_\phi)\,|\,|\alpha_i(w)|^{m_i}\geqslant\alpha\,\forall i=1,\ldots,r\}=\{w\in S\,|\,\alpha^{1/m_i}\leqslant|\alpha_i(w)|\leqslant t_\phi\forall_i\}$. Since Ω_0 is a compact subset of S, (1.6) implies that $W\Gamma/\Gamma$ is contained in a compact subset of G/Γ, thus proving the proposition in the case at hand.

Now let ξ be an admissible sequence of length $p\geqslant1$, say $\xi=((i_1,\lambda_1),\ldots,(i_p,\lambda_p))$, where $i_1,\ldots,i_p$ are distinct elements of $\{1,\ldots,r\}$ and $\lambda_1,\ldots,\lambda_p\in\mathbf{G}_{\mathbf{Q}}$ are such that $\lambda_{j-1}^{-1}\lambda_j\in\Lambda(\{i_1,\ldots,i_{j-1}\})$ for all $j=1,\ldots,p$, with $\lambda_0=e$, the identity element. For $j=1,\ldots,p$ let $I(j)=\{i_1,\ldots,i_j\}$. We first show that there exist compact subsets $K_1,\ldots,K_p$ of G such that for each $j=1,\ldots,p$ and $g\in W$ there exists a $k_j\in K_j$ such that $k_jg\lambda_j\in Q_{I(j)}$. We proceed by induction on j. We choose $K_1=K_0^{-1}$ where K_0 is a compact subset for which the contention of Lemma 1.5 holds for the choices $I=\phi$, $j=i_1$ and a and b as in the hypothesis of the Proposition. Since $d_{i_1}(g\lambda_1)\in[a,b]$ for all $g\in W$, the Lemma implies that for each $g\in W$ there exists a $k_1\in K_1$ such that $k_1g\lambda_1\in Q_{I(1)}$. Now suppose that compact subsets $K_1,\ldots,K_j$ have been found, satisfying the condition as above for some $1\leqslant j\leqslant p-1$. By Lemma 1.4 there exists a $c\in(0,1)$ such that $d_{i_{j+1}}(xh)\geqslant cd_{i_{j+1}}(h)$ for all $x\in K_j\cup K_j^{-1}$ and $h\in G$. Let K_0 be the compact subset for which the contention of Lemma 1.5 holds for the choices $I=I(j)$ and $j=i_{j+1}$ and ca and $c^{-1}b$ in the place of a and b. Put $K_{j+1}=K_0^{-1}K_j$. Now let $g\in W$. By our choice there exists a $k_j\in K_j$ such that $k_jg\lambda_j\in Q_{I(j)}$. Since $\lambda_j^{-1}\lambda_{j+1}\in Q_{I(j)}$ we get that $k_jg\lambda_{j+1}\in Q_{I(j)}$. Further, we have

$$ca\leqslant cd_{i_{j+1}}(g\lambda_{j+1})\leqslant d_{i_{j+1}}(k_jg\lambda_{j+1})\leqslant c^{-1}d_{i_{j+1}}(g\lambda_{j+1})\leqslant c^{-1}b.$$

Hence by Lemma 1.5 there exists a $k_0\in K_0$ such that $k_jg\lambda_{j+1}\in k_0Q_{I(j+1)}$. Thus we see that for $k_{j+1}=k_0^{-1}k_j$, $k_{j+1}g\lambda_{j+1}\in Q_{I(j+1)}$ as desired. Thus the inductive construction is complete.

that $W\Gamma/\Gamma \subset K_r^{-1} Q_I \lambda_r^{-1} \Gamma/\Gamma$, which is a compact subset. Now suppose that I is a proper subset. By (1.1) and (1.2) there exists a $\theta \in \Lambda(I(p))$ such that $k_p g \lambda_p \theta \in K\Omega(t_{I(p)})\Psi_{I(p)}$ say $k_p g \lambda_p \theta = kw\psi$ where $k \in K, w \in \Omega(t_{I(p)})$ and $\psi \in \Psi_{I(p)}$. Let $J = \{1, \ldots, r\} - I(p)$. Observe that for any $j \in J$, $(j, \lambda_p \theta) \in \mathscr{C}(\xi)$ and hence $d_j(k_p g \lambda_p \theta) \geqslant \beta$. Hence we get that

$$|\alpha_j(w)|^{m_j} = d_j(kw\psi) = d_j(k_p g \lambda_p \theta) \geqslant \beta.$$

Let

$$\begin{aligned}\Omega_0 &= \{w \in \Omega(t_{I(p)}) \,\|\, \alpha_j(w)|^{m_j} \geqslant \beta \quad \forall j \in J\} \\ &= \{w \in S_J \,|\, \beta^{1/m_j} \leqslant |\alpha_j(w)| \leqslant t_{I(p)}\}.\end{aligned}$$

Then Ω_0 is a compact subset of S_J and the above argument shows that for any $g \in W$ there exist a $k_p \in K_p$ and a $\theta \in \Lambda(I(p)) = (F_{I(p)}\Gamma)^{-1} = \Gamma F_{I(p)}^{-1}$ such that $k_p g \lambda_p \theta \in K\Omega_0 \Psi_{I(p)}$. Therefore

$$W \subset K_p^{-1} K \Omega_0 \Psi_{I(p)} F_{I(p)} \Gamma \lambda_p^{-1}. \tag{1.7}$$

Since $K_p^{-1} K \Omega_0 \Psi_{I(p)}$ is a compact subset of G and $F_{I(p)} \Gamma \lambda_p^{-1}$ is contained in a finite union of cosets of Γ, (1.7) implies that $W\Gamma/\Gamma$ is contained in a compact of G/Γ. This proves the Proposition.

PROPOSITION 1.8

Let ξ be an admissible sequence of length $p \geqslant 1$; say $\xi = ((i_1, \lambda_1), \ldots, (i_p, \lambda_p))$. Let α, a and b be positive real numbers and let W be the subset of G as in Proposition 1.3 for this data. Let $I = \{i_1, \ldots, i_p\}$. Then

$$W = \{g \in G \,|\, d_i(g\lambda_p\theta) \geqslant \alpha \forall i \notin I \text{ and } \theta \in \Lambda(I) \text{ and } a \leqslant d_i(g\lambda_p) \leqslant b \forall i \in I.\}$$

In particular, the set $W\Gamma/\Gamma$ is determined by I and $\Gamma\lambda_p$, in the sense that if $\xi' = ((i_1, \lambda'_1), \ldots, (i_p, \lambda'_p))$ is an admissible sequence and $\lambda'_p \in \Gamma\lambda_p$, then the corresponding set for ξ' is the same as $W\Gamma/\Gamma$.

Proof. For any $1 \leqslant j \leqslant p$ let $I(j) = \{i_1, \ldots, i_j\}$. Since, by admissibility of $\xi, \lambda_j^{-1}\lambda_{j+1} \in \Lambda(I(j)) \subset Q_{I(j)}$ for all $j = 1, \ldots, p-1$ we get that $\lambda_j^{-1}\lambda_p \in Q_{I(j)}$ for all j. Therefore if $i = i_j$ for some j then $d_i(g\lambda_j) = d_i(g\lambda_p)$. Also clearly $(i, \lambda) \in \mathscr{C}(\xi)$ if and only if $i \notin I$ and $\lambda = \lambda_p\theta$ for some $\theta \in \Lambda(I)$. The first part of the proposition is immediate from these two observations. The remaining part now follows from an obvious substitution argument.

2. More on the functions d_i

We follow the notation as before. For each $i = 1, \ldots, r$ we define a representation ρ_i of G as follows. Let $1 \leqslant i \leqslant r$. Let U_i be the unipotent radical of P_i and let u_i be the dimension of U_i. Let $\mathscr{G}$ be the Lie algebra of G. Let $V_i = \wedge^i \mathscr{G}$, the ith exterior power of $\mathscr{G}$. We define ρ_i as the ith exterior power representation of the adjoint representation of G over $\mathscr{G}$. We equip $\mathscr{G}$ with a AdK-invariant norm. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathscr{G}$ with respect to the norm. For any ℓ, this defines a canonical basis of $\wedge^l \mathscr{G}$,

ch V_i; we equip V_i with the norm, denoted by $\|\cdot\|$, making the basis into an thonormal basis. It is straightforward to verify that the norm is $\rho_i(K)$-invariant. t p_i be an element of norm 1 in the one-dimensional subspace of $V_i = \wedge^i \mathscr{G}$ rresponding to the Lie subalgebra of $\mathscr{G}$ associated to U_i, which is a u_i-dimensional bspace. A straightforward computation shows that

$$\rho_i(x)(p_i) = \alpha_i(x)^{m_i} p_i \quad \forall x \in P_i. \tag{2.1}$$

is implies that $d_i(x) = \|\rho_i(x)(p_i)\|$ for all $x \in P_i$. Since d_i and the norm are K-invariant d $G = KP_i$ we get that

$$d_i(g) = \|\rho_i(g)(p_i)\| \quad \forall g \in G. \tag{2.2}$$

We also note at this point that for $g \in G$, $\rho_i(g)(p_i) = p_i$ if and only if $g \in Q_i$. The 'if' rt follows from (2.1). Now let $g \in G$ be such that $\rho_i(g)(p_i) = p_i$. Then the definition ρ_i shows that the Lie subalgebra of U_i is $Ad\ g$-invariant. Since U_i is a connected e subgroup this implies that g normalizes U_i. But P_i is the normalizer of U_i (cf. [2]). ence $g \in P_i$. But then by (2.1) $\alpha_i(g) = 1$ which means that $g \in Q_i$.

OPOSITION 2.3

t $1 \leqslant i \leqslant r$ and let n_i be the dimension of V_i. Let $\{u_t\}$ be a unipotent one-parameter bgroup of G and let $g \in G$. Then $d_i^2(u_t g)$ is a polynomial in t of degree at most $2(n_i - 1)$. rther $d_i(u_t g)$ is constant (that is, independent of t) if and only if $g^{-1} u_t g \in Q_i$ for all $t \in \mathbf{R}$.

oof. Since $\{u_t\}$ is a unipotent one-parameter subgroup of G, $\{\rho_i(u_t)\}$ is a unipotent e-parameter group of linear transformations of V_i. By Jordan decomposition this plies that for any $v \in V_i$ the expansion of $\{\rho_i(u_t)(v)\}$ with respect to any basis has efficients which are polynomials in t of degree at most $(n_i - 1)$. Applying this to an honormal basis we see that for any $v \in V_i$, $\|\rho_i(u_t)(v)\|^2$ is a polynomial of degree most $2(n_i - 1)$. Given $g \in G$, choosing $v = \rho_i(g)p_i$ we see that $\|\rho_i(u_t g)(p_i)\|^2$ is a lynomial of degree at most $2(n_i - 1)$ and hence by (2.2) so is $d_i^2(u_t g)$.

Now let $g \in G$ be such that $d_i(u_t g)$ is constant in t. Then by (2.2), $\|\rho_i(u_t g)(p_i)\| =$ $(u_t)\rho_i(g)(p_i)\|$ is constant. For a unipotent one-parameter group of linear transfor- tions any orbit other than a fixed point is an unbounded subset of the vector ace. Therefore under the above condition $\rho_i(u_t)\rho_i(g)(p_i) = \rho_i(g)(p_i)$ for all $t \in \mathbf{R}$. Hence $g^{-1}u_t g)$ fixes p_i for all t. As noted before, this implies that $g^{-1}u_t g \in Q_i$ for all $t \in \mathbf{R}$. is proves the Proposition.

mma 2.4. *Let $1 \leqslant i \leqslant r$, $f \in \mathbf{G}_Q$ and $g \in G$ be given. Then for any $\delta > 0$ the set $\in \Gamma \,|\, d_i(g\gamma f) < \delta\}$ is finite.*

oof. Let $\mathscr{G}$ be equipped with the $\mathbf{Q}$-structure corresponding to the $\mathbf{Q}$-structure on Since $\mathbf{U}_i$ is an algebraic subgroup defined over $\mathbf{Q}$, the Lie subalgebra of $\mathscr{G}$ rresponding to U_i is a rational subspace (spanned, over $\mathbf{R}$, by rational elements) $\mathscr{G}$. The $\mathbf{Q}$-structure on $\mathscr{G}$ induces canonically a $\mathbf{Q}$-structure on $V_i = \wedge^i \mathscr{G}$ and is (the restriction of) a rational representation with respect to the $\mathbf{Q}$-structure. Also view of the preceding assertion p_i is a scalar multiple of a rational element, say $= tq_i$ where $t \in \mathbf{R}$ and q_i is rational. Since $f \in \mathbf{G}_Q$ we get that $\rho_i(f)(q_i)$ is rational

Since Γ is an arithmetic subgroup, this implies in turn that $\rho_i(\Gamma)\rho_i(f)q_i$ is a discrete subset of V_i. Since

$$\rho_i(g\Gamma f)(p_i) = \rho_i(g)\rho_i(\Gamma)\rho_i(f)(p_i) = t\rho_i(g)\rho_i(\Gamma)\rho(f)(q_i)$$

we get the $\rho_i(g\Gamma f)(p_i)$ is a discrete subset of V_i. In particular for any $\delta > 0$ there exist only finitely many $\gamma \in \Gamma$ such that $\|\rho_i(g\gamma f)p_i\| \geqslant \delta$. In view of (2.2), this implies the Lemma.

Lemma 2.5. *There exists a finite subset $\tilde{F}$ of $\mathbf{G}_Q$ such that for any admissible sequence ξ and any $(i, \lambda) \in supp\ \xi, \lambda \in \Gamma\tilde{F}$.*

Proof. If $((i_1, \lambda_1), \ldots, (i_p, \lambda_p))$ is an admissible sequence of length $p \geqslant 1$ then for all $j = 2, \ldots, p$ we have $\lambda_{j-1}^{-1}\lambda_j \in \Lambda(I(j-1))$, where $I(k) = \{i_1, \ldots, i_k\}$ for all k, and hence $\lambda_j \in \Lambda(\phi)\Lambda(I(1)) \ldots \Lambda(I(j-1))$. This shows that for any admissible sequence ξ and any $(i, \lambda) \in \operatorname{supp} \xi, \lambda$ is an element of a set of the form $\Lambda(\phi)\Lambda(I_1) \ldots \Lambda(I_j)$ where $j \in \{1, \ldots, r-1\}$ and $I_1, \ldots, I_j$ are subsets of $\{1, \ldots, r\}$ of cardinalities $1, \ldots, j$ respectively, such that $I_1 \subset I_2 \subset \cdots \subset I_j$. Since each $\Lambda(I)$, $I \subset \{1, \ldots, r\}$, is a finite union of cosets of the form Γf, $f \in \mathbf{G}_Q$ and Γ is an arithmetic lattice, it follows that each product $\Lambda(\phi)\Lambda(I_1) \ldots \Lambda(I_j)$ as above is a finite union of cosets of the form Γf, $f \in \mathbf{G}_Q$. Hence the preceding assertion implies that there are finitely many such cosets which together contain the supports of all admissible sequences. We can therefore choose a subset $\tilde{F}$ of $\mathbf{G}_Q$ for which the contention of the Lemma holds.

Lemma 2.6. *Let $1 \leqslant i \leqslant r$ and let $\{u_t\}$ be a unipotent one-parameter subgroup of G. Then the function v: $\mathbf{R} \to \mathbf{R}$ defined by*

$$v(t) = \sup\{d_i(u_t g)/d_i(g) \mid g \in G\}\ \forall t \in \mathbf{R}$$

is continuous.

Proof. Consider the function φ: $\mathbf{R} \times G \to \mathbf{R}$ defined by $\varphi(t, g) = d_i(u_t g)/d_i(g)$ for all $t \in \mathbf{R}$ and $g \in G$. Since $d_i(hp) = d_i(h)d_i(p)$ for all $h \in G$ and $p \in P_i$ we see that $\varphi(t, gp) = \varphi(t, g)$ for all $t \in \mathbf{R}$, $g \in G$ and $p \in P_i$. Hence we get a well-defined function $\tilde{\varphi}$: $\mathbf{R} \times G/P_i \to \mathbf{R}$ such that $\tilde{\varphi}(t, gP_i) = \varphi(t, g)$ for all $t \in \mathbf{R}$ and $g \in G$. Since φ is continuous so is $\tilde{\varphi}$. Also, clearly

$$v(t) = \sup\{\tilde{\varphi}(t, x) \mid x \in G/P_i\}.$$

Since $\tilde{\varphi}$ is continuous and G/P_i is compact, an elementary argument shows that the right hand side is a continuous function. This proves the lemma.

PROPOSITION 2.7

Let $1 \leqslant i \leqslant r$, let $\{u_t\}$ be a unipotent one-parameter subgroup of G and let $g \in G$. Let A

Proof. Let v: $\mathbf{R}\to\mathbf{R}$ be the function as in Lemma 2.6 for i and $\{u_t\}$ as above. By the Lemma there exists a neighbourhood Ω of 0 such that $v(t)<2$ for all $t\in\Omega$. By the definition of s there exist sequences $\{t_k\}$ in $[t_1,t_2]$ and $\{\lambda_k\}$ in A such that $t_k\to s$ and $d_i(u_{t_k}g\lambda_k)=\delta$ for all k. We may clearly assume that $t_k-s\in\Omega$ for all k. Then $d_i(u_sg\lambda_k)\leqslant v(s-t_k)d_i(u_{t_k}g\lambda_k)\leqslant 2\delta$ for all k. Since A is contained in finitely many cosets of the form Γf, by Lemma 2.4 this implies that $\{\lambda_k|k=1,2,\ldots\}$ is a finite set. Passing to a subsequence we may assume that $\lambda_k=\lambda$ for all k, where $\lambda\in A$. Then, since $t_k\to s$ and $d_i(u_{t_k}g\lambda)=\delta$ for all k, we get that $d_i(u_sg\lambda)=\delta$. This proves the Proposition.

3. Proof of Theorem 1

In this section we complete the proof of Theorem 1. We begin by recalling some properties of nonnegative polynomials and fixing some more notation.

For $m\in N$ let $\mathscr{P}_m$ denote the set of all nonnegative valued polynomials of degree at most m. We need the following simple properties of nonnegative polynomials (cf. [9] Lemma A.4 or [5] Lemmas 1.3 and 1.4).

Lemma 3.1. a) *For any $m\in\mathrm{N}$ and $\rho>0$ there exists a $\alpha>0$ such that the following holds: If $P\in\mathscr{P}_m$ is such that $P(1)<\alpha$ and $P(s)\geqslant 1$ for some $s\in[0,1]$ then there exists a $t\in[1,\rho]$ such that $P(t)=\alpha$.*

b) *For any $m\in\mathrm{N}$ and $\sigma>1$ there exist constants $\beta_1,\beta_2>0$ such that the following holds: If $P\in\mathscr{P}_m$, $P(s)\leqslant 1$ for all $s\in[0,1]$ and $P(1)=1$ then there exists a ℓ, $0\leqslant\ell\leqslant m$, such that $\beta_1\leqslant P(t)\leqslant\beta_2$ for all $t\in[\sigma^{2\ell+1},\sigma^{2\ell+2}]$.*

For the rest of the argument we fix some constants as follows. Let $\varepsilon>0$ be arbitrary (we shall later choose this to be as in Theorem 1). Let $\sigma>1$ be such that $(1-\sigma^{-1})^r>(1-\varepsilon)$ where r, as in § 1, is the **Q**-rank of G. We next choose $\tau>1$ such that $(\tau^{-1}-\sigma^{-1})^r\geqslant(1-\varepsilon)$. Let $m=2\max\{n_i-1\,|\,1\leqslant i\leqslant r\}$ and let $\rho>1$ be such that $(\rho-1)\leqslant(\tau-1)/\sigma^{2m+2}$. Let $\alpha\in(0,1)$ be such that the contention of Lemma 3.1 a) holds for these choices of m and ρ. Let $0<\beta_1<1<\beta_2$ be such that the contention of Lemma 3.1 b) holds for the choices of m and σ as above.

PROPOSITION 3.2

Let $\{u_t\}$ be a unipotent one-parameter subgroup of G and let $g\in G$. Let ξ be an admissible sequence of length $p\geqslant 0$. Let $s\geqslant 0$ and $\chi>0$ be such that for any $(i,\lambda)\in\mathscr{C}(\xi)$ there exists a $t\in[0,s]$ such that $d_i^2(u_tg\lambda)\geqslant\chi$. Then at least one of the following conditions holds:

(i) *there exists a $s'\in(s,\tau s)$ such that for all $(i,\lambda)\in\mathscr{C}(\xi)$ and $t\in[s,s')$*

$$d_i^2(u_tg\lambda)>\chi\alpha/2$$

(ii) *there exist s_0, $s_1\in[s,\tau s]$ such that $(s_1-s)=\sigma(s_0-s)$ and the following conditions are satisfied:*

First suppose that $\mathscr{F}$ is empty. Consider the set

$$E=\{t\in[s,\tau s)\,|\,d_i^2(u_t g\lambda)>\chi\alpha/2\;\forall(i,\lambda)\in\mathscr{C}(\xi)\}.$$

If $E=[s,\tau s)$ then condition i) of the Proposition holds for $s'=\tau s$. Now suppose that E is a proper subset of $[s,\tau s)$. Let $s'=\inf\{t\,|\,t\in[s,\tau s)-E\}$. Then by Lemma 2.5 and Proposition 2.7, there exists a $(i,\lambda)\in\mathscr{C}(\xi)$ such that $d_i^2(u_s' g\lambda)=\chi\alpha/2$. Hence $s'\in[s,\tau s)-E$. On the other hand, since $\mathscr{F}$ is empty $s\in E$. In particular $s'>s$. Clearly condition i) of the Proposition holds for this s'.

Next suppose that $\mathscr{F}$ is nonempty. By Lemmas 2.4 and 2.5 $\mathscr{F}$ is a finite set. By hypothesis for any $(i,\lambda)\in\mathscr{F}\subset\mathscr{C}(\xi)$ there exists a $t\in[0,s]$ such that $d_i^2(u_t g\lambda)\geqslant\chi$ and hence by Lemma 3.1 a), applied to the polynomial $t\mapsto d_i^2(u_{st}g\lambda)/\chi$, which is of degree $2(n_i-1)\leqslant m$ (cf. Proposition 2.3), there exists a $t\in[s,\rho s]$ such that $d_i^2(u_t g\lambda)=\chi\alpha$. For each $(i,\lambda)\in\mathscr{C}(\xi)$ let $t(i,\lambda)=\inf\{t\in[s,\rho s]\,|\,d_i^2(u_t g\lambda)=\chi\alpha\}$ and let $y=\max\{t(i,\lambda)\,|\,(i,\lambda)\in\mathscr{F}\}$. Let $(j,\mu)\in\mathscr{F}$ be such that $t(j,\mu)=y$. We note that

$$d_j^2(u_y g\mu)=\chi\alpha\geqslant 2d_j^2(u_s g\mu). \tag{3.3}$$

Now observe that $d_j^2(u_t g\mu)\leqslant\chi\alpha$ for all $t\in[s,y]$ and $d_j^2(u_y g\mu)=\chi\alpha$. Hence by Lemma 3.1 b), applied to the polynomial $t\mapsto d_j^2(u_{s+(y-s)t}g\mu)/\chi\alpha$, there exists a $\ell, 0\leqslant\ell\leqslant m$, such that

$$\chi\alpha\beta_1\leqslant d_j^2(u_t g\mu)\leqslant\chi\alpha\beta_2\ldots\;\forall t\in[s_0,s_1] \tag{3.4}$$

where

$$s_0=s+\sigma^{2\ell+1}(y-s)\text{ and }s_1=s+\sigma^{2\ell+2}(y-s).$$

Observe that $s\leqslant s_0\leqslant s_1\leqslant s+\sigma^{2m+2}(\rho-1)s\leqslant\tau s$. Also clearly $(s_1-s)=\sigma(s_0-s)$. We next verify conditions ii) for these choices of s_0 and s_1. We see that for $(i,\lambda)\in\mathscr{C}(\xi)$, $d_i^2(u_s g\lambda)>\chi\alpha/2$ if $(i,\lambda)\notin\mathscr{F}$ and $d_i^2(u_{t(i,\lambda)}g\lambda)=\chi\alpha$ if $(i,\lambda)\in\mathscr{F}$; since $s\leqslant t(i,\lambda)\leqslant y<s_0$, this shows that condition ii) (a) holds. Condition ii) (b) follows from (3.3) and (3.4). This proves the Proposition.

PROPOSITION 3.5

Let $\{u_t\}$ be a unipotent one-parameter subgroup of G. Let ξ be an admissible sequence of length $p\geqslant 0$. Let $g\in G$, $s\geqslant 0$ and $\chi'\geqslant\chi>0$ be such that for any $(i,\lambda)\in\mathscr{C}(\xi)$ there exists a $t\in[0,s]$ such that $d_i^2(u_t g\lambda)\geqslant\chi$ and for any $(i,\lambda)\in\operatorname{supp}\xi$, $\chi\beta_1\leqslant d_i^2(u_t g\lambda)\leqslant\chi'\beta_2$ for all $t\in[s,\sigma s]$. For any admissible sequence ζ extending ξ, say of length q, let

$$X(\zeta)=\{t\in[s,\sigma s]\,|\,d_i^2(u_t g\lambda)\geqslant\chi(\alpha/2)^{q-p+1}\quad\forall(i,\lambda)\in\mathscr{C}(\zeta)\text{ and }$$
$$(\alpha/2^{q-p})\chi\beta_1\leqslant d_i^2(u_t g\lambda)\leqslant\chi'\beta_2\quad\forall(i,\lambda)\in\operatorname{supp}\xi\}$$

and let

$$X=\cup_\zeta X(\zeta)$$

olds for all admissible sequences of length $\geqslant p+1$, for all $g\in G$, $s\geqslant 0$ and $\chi>0$ atisfying the conditions in the hypothesis, and let an admissible sequence ξ of length , $g\in G$, $s\geqslant 0$ and $\chi>0$ be given, satisfying the conditions in the hypothesis. Let X e the set as in the statement of the Proposition, for this data.

We first show that for any $x\in[s,\tau^{-1}\sigma s]$ there exists a $x'\in(x,\tau x]$ for which either $x,x')\subset X$ or the following conditions are satisfied:

$$\ell(X\cap[x,x'])\geqslant(1-\sigma^{-1})(\tau^{-1}-\sigma^{-1})^{r-p-1}(x'-x) \tag{3.6}$$

nd there exist $(j,\mu)\in\mathscr{C}(\xi)$ and $y\in[x,x']$ such that

$$d_j^2(u_y g\mu)\geqslant 2d_j^2(u_x g\mu) \tag{3.7}$$

.et $x\in[s,\tau^{-1}\sigma s]$ be given. We apply Proposition 3.2 with x in the place of s, the equisite conditions being satisfied since $x\geqslant s$. Suppose Condition i), as in the onclusion of that Proposition, holds. Then there exists a $x'\in(x,\tau x)$ such that for all $,\lambda)\in\mathscr{C}(\xi)$ and $t\in[x,x')$, $d_i^2(u_t g\lambda)\geqslant\chi\alpha/2$. We also see that $[x,\tau x]\subset[s,\sigma s]$ and hence $\beta_1\leqslant d_i^2(u_t g\lambda)\leqslant\chi'\beta_2$ for all $(i,\lambda)\in\operatorname{supp}\,\xi$ and $t\in[x,\tau x]$. The two assertions imply hat $[x,x')\subset X(\xi)\subset X$ and hence we are through in this case. Next suppose that Condition ii) (of Proposition 3.2) holds. Thus there exist $s_0,s_1\in[x,\tau x]$ such that $_1-x)=\sigma(s_0-x)$ and the following conditions are satisfied: a) for any $(i,\lambda)\in\mathscr{C}(\xi)$ here exists a $t\in[x,s_0]$ such that $d_i^2(u_t g\lambda)\geqslant\chi\alpha/2$ and b) there exists a $(j,\mu)\in\mathscr{C}(\xi)$ such hat $\chi\alpha\beta_1\leqslant d_j^2(u_t g\mu)\leqslant\chi\alpha\beta_2$ for all $t\in[s_0,s_1]$ and $d_j^2(u_y g\mu)\geqslant 2d_j^2(u_x g\mu)$ for some $\in[x,s_0]$. Let η be the admissible sequence of length $p+1$ extending ξ and containing $,\mu)$ (as in condition (b)) as the last term. Then we see that the conditions in the ypothesis of the present proposition are satisfied for η, in the place of ξ, with $_x g, s_0-x$, and $\chi\alpha/2$ in the place of g, s and χ respectively: condition a) above implies hat for any $(i,\lambda)\in\mathscr{C}(\eta)$ there exists a $t\in[x,s_0]$ such that $d_i^2(u_{(t-x)}u_x g\lambda)\geqslant\chi\alpha/2$. For all $,\lambda)\in\operatorname{supp}\,\xi$ we have

$$d_i^2(u_{t-x}u_x g\lambda)=d_i^2(u_t g\lambda)\in[\chi\beta_1,\chi'\beta_2]\subset[\chi(\alpha/2)\beta_1,\chi'\beta_2]$$

or all $t\in[s,\sigma s]$ and, in particular, whenever $(t-x)\in[s_0-x,\sigma(s_0-x)]$, since $\sigma(s_0-x)=$ $_1-x$ and $s_0,s_1\in[x,\tau x]\subset[s,\sigma s]$; also $d_j^2(u_t g\mu)\in[\chi\alpha\beta_1,\chi\alpha\beta_2]\subset[\chi(\alpha/2)\beta_1,\chi'\beta_2]$. Thus ve have verified the conditions in the hypothesis for the choices as above. Since η is f length $p+1$, by the induction hypothesis the assertion of the Proposition holds or η. For any admissible sequence ζ let $X'(\zeta)$ be the set corresponding to $X(\zeta)$ as in he proposition with respect to the choices as above. Let X' be the union of $X'(\zeta)$ ver all admissible sequences extending η. Then we have

$$\ell(X')\geqslant(\tau^{-1}-\sigma^{-1})^{r-p-1}(\sigma-1)(s_0-x) \tag{3.8}$$

t is straightforward to verify by substitution that for any admissible sequence ζ xtending η and $t\in X'(\zeta)$, $x+t\in X(\zeta)\cap[s_0,s_1]$; recall for this purpose that $[s_0-x,$ $(s_0-x)]=[s_0-x,\ s_1-x]\subset[s-x,\ \sigma s-x]$. Hence by (3.8) we get that

Also by condition b) above there exists a $y \in [x, s_0] \subset [x, x']$ such that (3.7) holds. Thus we have produced a x' for which (3.6) and (3.7) hold.

To complete the proof we construct a finite sequence $x_0, x_1, \ldots, x_n$ in $[s, \sigma s]$ as follows. We choose $x_0 = s$. Let $k \geqslant 0$ and suppose that $x_0, \ldots, x_k$ have been chosen. If $x_k \leqslant \tau^{-1}\sigma s$ then we choose $x_{k+1} \in [x_k, \tau x_k]$ as follows: If there exists $x' \in (x_k, \sigma s)$ such that $[x_k, x') \subset X$ then we choose x_{k+1} to be such that $[x_k, x_{k+1}) \subset X$ but $[x_k, x'')$ is not contained in X for any $x'' > x_{k+1}$. If there does not exist any $x' > x_k$ with $[x_k, x') \subset X$ then, as $x_k \in [s, \tau^{-1}\sigma s]$, by what we proved above (see (3.6)) there exists a $x_{k+1} \in (x_k, \tau x_k]$ such that

$$\ell(X \cap [x_k, x_{k+1}]) \geqslant (1 - \sigma^{-1})(\tau^{-1} - \sigma^{-1})^{r-p-1}(x_{k+1} - x_k) \tag{3.9}$$

and there exist $(j, \mu) \in \mathscr{C}(\xi)$ and $y \in [x_k, x_{k+1}]$ such that

$$d_j^2(u_y g\mu) \geqslant 2d_j^2(u_{x_k} g\mu). \tag{3.10}$$

Observe that since $x_k \leqslant \tau^{-1}\sigma s$, $x_{k+1} \leqslant \sigma s$. Lastly, if $x_k > \tau^{-1}\sigma s$ we terminate the sequence, setting $n = k$.

We show that the sequence as defined above does terminate in finitely many steps. For this purpose observe that if for some $k \geqslant 0$, $[x_k, x_{k+1}) \subset X$ then $[x_{k+1}, x')$ is not contained in X for any $x' > x_{k+1}$. In view of this, to show that the sequence terminates it is enough to show that there exists a $c > 0$ such that $x_{k+1} - x_k \geqslant c$ for any $k \geqslant 0$ such that $[x_k, x_{k+1})$ is not contained in X. In view of Lemma 2.6 there exists a $c > 0$ such that if for some $i \in \{1, \ldots, r\}$, $h \in G$ and $t \geqslant 0$, $d_i(u_t h)/d_i(h) \geqslant \sqrt{2}$ then $t \geqslant c$. Recall that when $[x_k, x_{k+1})$ is not contained in X there exist $(j, \mu) \in \mathscr{C}(\xi)$ and $y \in [x_k, x_{k+1}]$ such that (3.7) holds and in that case, by the above observation, $y - x_k \geqslant c$ and in turn $x_{k+1} - x_k \geqslant c$, as desired. Hence the sequence indeed terminates (in at most $2(\tau^{-1}\sigma - 1)s/c$ steps!) at a $x_n > \tau^{-1}\sigma s$.

Now we have

$$\ell(X) \geqslant \sum_{k=0}^{n-1} \ell(X \cap [x_k, x_{k+1}]) \geqslant (1 - \sigma^{-1})(\tau^{-1} - \sigma^{-1})^{r-p-1}(x_n - x_0),$$

by (3.9). Since $(x_n - x_0) > (\tau^{-1}\sigma s - s) = \sigma(\tau^{-1} - \sigma^{-1})s$, this yields that

$$\ell(X) \geqslant (\sigma - 1)(\tau^{-1} - \sigma^{-1})^{r-p} s$$

thus proving the Proposition.

Proof of Theorem 1. Let $F \subset \mathbf{G}_{\mathbf{Q}}$ be a finite subset such that $\Lambda(\phi) = \Gamma F$ (cf. (1.2)). Now let $\varepsilon > 0$ and $\theta > 0$ be as in the hypothesis of the Theorem and $\sigma > 1$ such that $(1 - \sigma^{-1})^r > (1 - \varepsilon)$. Let $\tau > 1$, $\rho > 1$, $\alpha \in (0, 1)$ and $0 < \beta_1 < 1 < \beta_2$ be the constants chosen as in the beginning of the section starting with σ. For any admissible sequence ζ of length q let

$$W(\zeta) = \{g \in G \mid d_i^2(g\lambda) \geqslant \theta(\alpha/2)^{q+1}\ \forall (i, \lambda) \in \mathscr{C}(\zeta) \text{ and }$$
$$(\alpha/2)^q \theta\beta_1 \leqslant d_i^2(g\lambda) \leqslant \theta\beta_2\ \forall (i, \lambda) \in \operatorname{supp}\zeta\}$$

and let

where the union is taken over all admissible sequences ζ. By Proposition 1.8 there are only finitely many distinct subsets involved in the union and by Proposition 1.3 each of them is compact. Hence C is a compact subset of G/Γ. We shall show that the contention of the Theorem holds for the compact set C and σ as above.

Let a unipotent one-parameter subgroup $\{u_t\}$ *in* $G, g \in G$ and $T \geqslant 0$ be given. For any admissible sequence ζ of length q let

$$X(\zeta) = \{t \in [T, \sigma T] \mid d_i^2(u_t g\lambda) \geqslant \theta(\alpha/2)^{q+1}\, \forall (i, \lambda) \in \mathscr{C}(\zeta) \text{ and}$$
$$(\alpha/2)^q \theta\beta_1 \leqslant d_i^2(u_t g\lambda) \geqslant \theta\beta_2\, \forall (i, \lambda) \in \operatorname{supp} \zeta\}$$

and let

$$X = \cup_\zeta X(\zeta)$$

the union being taken over all admissible sequences ζ. Applying Proposition 3.5 to the empty sequence ϕ, with $s = T$ and $\chi = \chi' = \theta^2$ we see that either there exists a $(i, \lambda) \in \mathscr{C}(\phi)$ such that $d_i(u_t g\lambda) < \theta$ for all $t \in [0, T]$ or.

$$\ell(X) \geqslant (\tau^{-1} - \sigma^{-1})^r(\sigma - 1)\, T.$$

Observe that if $t \in X$ then $u_t g\Gamma \in C$. Recall also that by choice $(\tau^{-1} - \sigma^{-1})^r \geqslant (1 - \varepsilon)$ and that for $i \in \{1, \ldots, r\}$, $(i, \lambda) \in \mathscr{C}(\phi)$ if and only if $\lambda \in \Lambda(\phi) = \Gamma F$. Hence the above conclusion implies the assertion in the theorem, that either

$$\ell(\{t \in [T, \sigma T] u_t g\Gamma \in C\}) \geqslant (1 - \varepsilon)(\sigma - 1)\, T$$

or these exist $\lambda \in \Gamma F$ and $i \in \{1, \ldots, r\}$ such that $d_i(u_t g\lambda) < \theta$ for all $t \in [0, T]$.

4. Proofs of the other theorems

We shall now deduce the other theorems stated in the introduction. We follow the same notation as before.

Proof of Theorem 2. Let $\varepsilon > 0$ and $\theta > 0$ be given and let C be a compact subset of G/Γ for which the contention of Theorem 1 holds for $\varepsilon/2$ and θ in the place of ε and θ respectively. Let $\{u_t\}$ be a unipotent one-parameter subgroup of G and let $g \in G$. Let $\sigma > 1$ be such that $(1 - \sigma^{-1})^r > (1 - \varepsilon/2)$. Then by Theorem 1 for any $T \geqslant 0$ either there exist $j \in \{1, \ldots, r\}$ and $\mu \in \Gamma F$ such that $d_j(u_t g\mu) < \theta$ for all $t \in [0, \sigma^{-1} T]$ or

$$\ell(\{t \in [\sigma^{-1} T, T] \mid u_t g\Gamma \in C\}) \geqslant (1 - \varepsilon/2)(\sigma - 1)\sigma^{-1}\, T > (1 - \varepsilon)\, T.$$

Hence if the first condition in the conclusion of Theorem 2 does not hold then for each $T \geqslant 0$ there exist $j \in \{1, \ldots, r\}$ and $\mu \in \Gamma F$ such that $d_j(u_t g\mu) < \theta$ for all $t \in [0, \sigma^{-1} T]$. By Lemma 2.4 the set

implies, by the second part of Proposition 2.3 that $\lambda^{-1}g^{-1}u_t g\lambda\in Q_i$, or equivalent $g^{-1}u_t g\in\lambda Q_i\lambda^{-1}$ for all $t\in\mathbf{R}$. This proves the theorem.

Proof of Theorem 3. Let F be a finite subset of $\mathbf{G_Q}$ and C be a compact subset G/Γ such that the contention of Theorem 2 holds, for some choice of $\varepsilon>0$ and $\theta>$ Let V and $\{x_k\}$, satisfying the conditions as in the statement of the Theorem, a $g\in G$ be given. If $\{u_t\}$ be any one-parameter subgroup of V and $k\geqslant 1$ then by Theorem either there exists a $t\geqslant 0$ such that $u_t x_k g\Gamma\in C$ or there exist $i\in\{1,\dots,r\}$ and $\lambda\in\Gamma$ such that $g^{-1}x_k^{-1}u_t x_k g\in\lambda Q_i\lambda^{-1}$ for all $t\in\mathbf{R}$ and $d_i(x_k g\lambda)<\theta$. Let $k\geqslant 1$ be such th $C\cap Vx_k g\Gamma/\Gamma$ is empty. Then by the last observation every one-parameter subgrou of V is contained in one of the subgroups $x_k g\mu Q_j\mu^{-1}g^{-1}x_k^{-1}$ for some $1\leqslant j\leqslant r$ a $\mu\in\Gamma F$ such that $d_j(x_k g\mu)<\theta$. Since the latter is a countable family of subgroups a V is an analytic subgroup, this implies that there exist $i\in\{1,\dots,r\}$ and $\lambda\in\Gamma F$ su that $V\subset x_k g\lambda Q_i\lambda^{-1}g^{-1}x_k^{-1}$ and $d_i(x_k g\lambda)<\theta$. Since $Q_i\subset P_i$ and V is in gener position we also get that $x_k g\lambda\in P_i$. Thus for any $k\geqslant 1$ such that $C\cap Vx_k g\Gamma/\Gamma=$ there exist $i\in\{1,\dots,r\}$ and a $\lambda\in\Gamma F$ such that $x_k g\lambda\in P_i$ and $d_i(x_k g\lambda)<\theta$.

Now suppose that the assertion in the Theorem does not hold for the compact s C as above. Then by the above observation there exist a subsequence of $\{x_k\}$, s $\{y_k\}$, $i\in\{1,\dots,r\}$ and a sequence $\{\lambda_k\}$ in ΓF such that $y_k g\lambda_k\in P_i$ and $d_i(y_k g\lambda_k)<\theta$ f all k. Since $y_k\in P_0\subset P_i$ and $y_k g\lambda_k\in P_i$ we get that $d_i(y_k g\lambda_k)=d_i(y_k)d_i(g\lambda_k)$ for all Now while $d_i(y_k g\lambda_k)<\theta$ for all k, since $\{y_k\}$ is a subsequence of $\{x_k\}$, by hypothes $d_i(y_k)\to\infty$. Therefore we get that $d_i(g\lambda_k)\to 0$ as $k\to\infty$. But by Lemma 2.4 this impossible since $\{\lambda_k\}$ is contained in ΓF which is finite union of cosets of the for $\Gamma f, f\in\mathbf{G_Q}$.

Proof of Theorem 4. Let Γ be a lattice in $SL(3,\mathbf{R})$. If $SL(3,\mathbf{R})/\Gamma$ is compact then t assertion is obvious. We shall therefore assume that G/Γ is noncompact. Then by t arithmeticity theorem (cf. [11]) there exists an algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$ su that $SL(3,\mathbf{R})$ is Lie isomorphic to $\mathbf{G_R}$ and under the isomorphism Γ corresponds an arithmetic lattice in $\mathbf{G_R}$ with respect to the $\mathbf{Q}$-structure on $\mathbf{G}$. We now follow t notation as before with respect to this $\mathbf{G}$ and identify $G=\mathbf{G_R}$ with $SL(3,\mathbf{R})$ via isomorphism. We note that since G/Γ is noncompact the $\mathbf{Q}$-rank r of $\mathbf{G}$ is at lea 1. On the other hand clearly $r\leqslant 2$, which is the $\mathbf{R}$-rank of $SL(3,\mathbf{R})$. Now let F be finite subset of $\mathbf{G_Q}$ and C be a compact subset of G/Γ such that the contentions Theorems 2 and 3 hold (the former for some choices of $\varepsilon>0$ and $\theta>0$). Let $g\in G$ given. Suppose that one of the sets $\{t\geqslant 0\mid v_1(t)g\Gamma\in C\}$ and $\{t\leqslant 0\mid v_1(t)g\Gamma\in C\}$ bounded. Then by Theorem 2, applied to either $\{v_1(t)\}$ or $\{v_1(-t)\}$ in the place $\{u_t\}$, we get that there exist an $i\in\{1,r\}$ and a $\lambda\in\Gamma F$ such that $g^{-1}v_1(t)g\in\lambda Q_i\lambda^{-1}$ f all $t\in\mathbf{R}$. Put $P=\lambda P_i\lambda^{-1}$. Let L be the closed subgroup generated by all unipote elements in P. Then we have $g^{-1}v_1(t)g\in L$ for all $t\in\mathbf{R}$. Also L is the group of $\mathbf{R}$-elemen of an algebraic subgroup $\mathbf{L}$ which is defined over $\mathbf{Q}$ and has no character defin over $\mathbf{Q}$. This implies that $L\Gamma$ is closed and $L\cap\Gamma$ is a lattice in L (cf. [4] §2). T shows that condition a) as in the definition of Condition (*) holds for the set for t set C (as above).

Let P_0 be the minimal $\mathbf{Q}$-parabolic subgroup of G as before. It is easy to see th $N(V_1)$ is contained in a Borel subgroup, specifically the group of upper triangul matrices. Hence there exists a $h\in G$ such that $hN(V_1)h^{-1}\subset P_0$. We shall show th

holds for the compact set $C \cup h^{-1}C$ (in the place of C in the definition). Let $\{f(t)\}_{t\geq 0}$ be a curve in $N(V_1)$ such that $|\det f(t)|W| \to \infty$ as $t \to \infty$ for every proper nonzero $N(V_1)$-invariant subspace. Put $V = hV_1h^{-1}$ and $\varphi(t) = hf(t)h^{-1}$ for all $t \geq 0$. Then $\{\varphi(t)\}_{t\geq 0}$ is a curve in $N(V) \subset P_0$ and $|\det \varphi(t)|W| \to \infty$ for every proper nonzero $N(V)$-invariant subspace. We shall deduce from this that $d_i(\varphi(t)) \to \infty$ as $t \to \infty$ for any $i \in \{1, r\}$. We first assume this and complete the proof. By Theorem 3 it yields that $C \cap V\varphi(t)hg\Gamma/\Gamma$ is nonempty for all large t. Substituting for V and $\varphi(t)$ we get that $C \cap hV_1 f(t)g\Gamma/\Gamma$ is nonempty for all large t, or equivalently, $h^{-1}C \cap V_1 f(t)g\Gamma/\Gamma$ is nonempty for all large t. This shows that condition b) holds for the compact set $h^{-1}C$, as desired.

It remains to prove that $d_i(\varphi(t)) \to \infty$ as $t \to \infty$ for any $i \in \{1, r\}$. Let $i \in \{1, r\}$ be given. First suppose that P_i is a maximal $\mathbf{R}$-parabolic subgroup. Then there exists a subspace W of $\mathbf{R}^3$ such that

$$P_i = \{g \in G \mid g(W_i) = W_i\}.$$

Further it is easy to see that in this case $d_i(x) = |\det x|W_i|^2$ for all $x \in P_i$. Since $|\det \varphi(t)|W| \to \infty$ for every proper nonzero $N(V)$-invariant subspace and $N(V) \subset P_0 \subset P_i$, this yields that $d_i(\varphi(t)) \to \infty$ as $t \to \infty$. Now suppose that P_i is not a maximal $\mathbf{R}$-parabolic subgroup. Since $\mathbf{G}$ has $\mathbf{R}$-rank 2, this implies that $\mathbf{P}_i$ is a minimal $\mathbf{R}$-parabolic subgroup. In turn we get $r = 1$, $i = 1$ and $P_0 = P_1$ and they are conjugate to the subgroup B consisting of upper triangular matrices; in fact $P_1 = hBh^{-1}$, since $h^{-1}P_1h$ has to be the Borel subgroup containing V_1. Using this we see that for all $t \geq 0$, $d_1(\varphi(t)) = (a_1(t)/a_3(t))^2 = a_1^4(t)a_2^2(t)$, where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are the diagonal entries of $f(t)$. Since $|\det f(t)|W| \to \infty$ for any $N(V_1)$-invariant proper non-zero subgroup, and $N(V_1) \subset B_1$, we get that $a_1^2(t) \to \infty$ and $a_1^2(t)a_2^2(t) \to \infty$ as $t \to \infty$. Hence $d_1(\varphi(t)) \to \infty$ as sought to be proved. This proves the Theorem.

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On the ratio of the maximum term and the maximum modulus of the sum of two entire functions

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Abstract. Some results on the sum of two entire functions pertaining to the ratio of the maximum term and the maximum modulus are proved.

Keywords. Entire functions; maximum term; maximum modulus; rank.

1. Introduction

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function. As usual, let $M(r,f)$, $\mu(r,f)$, and $v(r,f)$ denote the maximum modulus, the maximum term, and the rank of the maximum term of $f(z)$ respectively. Let $M_k (0 \leqslant k \leqslant 1)$ denote the class of all entire functions for which

$$\limsup_{r\to\infty} \frac{\mu(r,f)}{M(r,f)} = k.$$

The set of all entire functions of finite order is closed under addition. Now the question is, given any two transcendental entire functions belonging to M_k, does the sum, if it is transcendental also belong to M_k? (Polynomials are excluded, since polynomials are in M_1).

We shall answer this question in the negative. Actually, we shall show that, given a function of a class (even M_1), there is always another function of the same class such that their sum is not in the class. If the polynomials are also included, then the answer to the above question is immediate when $k \neq 1$. We shall also show that the sum of an entire function and its derivative, and the function are in the same class for certain entire functions, when r is allowed to tend to infinity over a set of infinite measure.

A result in this direction about the sum of two entire functions can be found

2. Results and Proofs

We shall state the results mentioned in § 1 in precise terms, and prove them.

Theorem 1. *Given an entire function $f_1(z) \in M_k (0 \leqslant k < 1)$ there exists another entire function $f_2(z) \in M_k$ such that $f_1(z) + f_2(z)$ is a transcendental entire function belonging to M_1.*

Theorem 2. *Given a transcendental entire function $f_1(z) \in M_1$ there exists another entire function $f_2(z) \in M_1$ such that $f_1(z) + f_2(z) \in M_k$, $k \neq 1$.*

We shall prove something more than merely that $f(z)$ and $f(z) + f^{(1)}(z)$ belong to the same class for certain entire functions, where in the limit superior, $r \to \infty$ over a set of infinite measure.

Theorem 3. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function for which*

$$\liminf_{r \to \infty} \left| \frac{a_n^2}{a_{n-1} a_{n+1}} \right| > 1,$$

then

$$\frac{\mu(r, f)}{M(r, f)} \sim \frac{\mu(r, f + f^{(1)})}{M(r, f + f^{(1)})} \tag{1}$$

as $r \to \infty$ over a set of infinite measure.

Proof of Theorem 1. Let

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Let

$$b_n = \begin{cases} [|a_n|^{-1}] + 1 & \text{if } a_n \neq 0 \\ 2 & \text{if } a_n = 0, \end{cases}$$

and

$$c_n = b_0 b_1 \dots b_n,$$

where $[|a_n|^{-1}]$ denotes the integral part of $|a_n|^{-1}$. Define

$$\tilde{f}_2(z) = \sum_{n=0}^{\infty} \frac{z^n}{c_0! c_1! \dots c_n!}.$$

For all r satisfying the inequalities $c_n! \leqslant r < c_{n+1}!$, we have

$$\begin{aligned} M(r, \tilde{f}_2) = \sum_{i=0}^{\infty} \frac{r^i}{c_0! c_1! \dots c_i!} &< \mu(r, \tilde{f}_2) \left(\dots + \frac{c_{n-1}!}{c_{n+1}!} + 3 + \frac{c_{n+1}!}{c_{n+2}!} + \dots \right) \\ &\leqslant \mu(r, \tilde{f}_2)(3 + B(n)), \end{aligned} \tag{2}$$

where $B(r) \to 0$ as $r \to \infty$. And for $r = c_n!(c_{n+1}!/c_n!)^{1/2}$, we have

where $c(r)\to 0$ as $r\to\infty$. From (3), it follows that $\tilde{f}_2(z)\in M_1$, and from (2), we have

$$\liminf_{r\to\infty}\frac{\mu(r,\tilde{f}_2)}{M(r,\tilde{f}_2)}\geqslant\frac{1}{3}.$$

Again for all r satisfying the inequalities $c_n!\leqslant r<c_{n+1}!$, we have

$$\frac{\mu(r,f_1)}{\mu(r,\tilde{f}_2)}=\frac{|a_n|r^n}{(c_0!c_1!\dots c_n!)^{-1}r^n}$$
$$>\frac{c_0!c_1!\dots c_n!}{c_n}.$$

This inequality yields

$$\liminf_{r\to\infty}\frac{\mu(r,f_1)}{\mu(r,\tilde{f}_2)}=\infty.$$

So

$$\liminf_{r\to\infty}\frac{\mu(r,f_1)}{M(r,\tilde{f}_2)}\geqslant\liminf_{r\to\infty}\frac{\mu(r,f_1)}{\mu(r,\tilde{f}_2)}\liminf_{r\to\infty}\frac{\mu(r,\tilde{f}_2)}{M(r,\tilde{f}_2)}=\infty.$$

Hence

$$M(r,f_1)\sim M(r,f_1+\tilde{f}_2)$$

as $r\to\infty$. Since

$$\mu(r,\tilde{f}_2)=o(\mu(r,f_1))$$

as $r\to\infty$, we have

$$\mu(r,f_1)\sim\mu(r,f_1+\tilde{f}_2)$$

as $r\to\infty$. Hence $f_1(z)+\tilde{f}_2(z)\in M_k$, $k\neq 1$. Clearly $\tilde{f}_2(z)\in M_1$. So, as $-f_1(z)\in M_k$, $k\neq 1$, the theorem is proved, and $f_1+\tilde{f}_2$ is our f_2. □

Proof of Theorem 2. Let

$$f_1(z)=\sum_{n=0}^{\infty}a_nz^n.$$

Since $f_1(z)\in M_1$ there is an unbounded set $S\subset\mathbb{R}_+$, the set of positive real numbers, such that

$$\frac{\mu(r,f_1)}{M(r,f_1)}\to 1$$

as $r\to\infty$ over S. Let $\{r_t\}_{t=1}^{\infty}$ be an unbounded sequence of S. For $t=1,2,\dots$, let

$$\nu(r_t,f_1)=n_t,$$

and

$$[|a_{\nu(r_t,f_1)}|^{-1}]=N_t.$$

generality, we can assume the following about the sequence $\{r_t\}_{t=1}^{\infty}$. For all $t=1,2,\ldots,N_t \geqslant \max(r_t, 2n_t)$; and $n_{t+1} \geqslant n_t + [\log n_t] + 3$, $N_{t+1} \geqslant N_t^2$. Let

$$T=\{n_t+h_t; t=1,2,\ldots,h_t=1,2,\ldots,[\log n_t]\},$$

$$b_{n_t+h_t}=\frac{1}{N_t+h_t!},$$

and

$$c_n=\begin{cases} b_n & \text{if } n\in T \\ 0 & \text{if } \mathbb{N}\, U\{0\}\setminus T, \end{cases}$$

where $\mathbb{N}$ denotes the set of all natural numbers. Define

$$f_2(z)=\sum_{n=0}^{\infty}(c_n-a_n)z^n.$$

Clearly $f_2(z)$ is an entire function $f_2(z)$. Since $c_{\nu(r_t,f_1)}=0$, we see that

$$\mu(r_t,f_2)\sim\mu(r_t,f_1) \tag{4}$$

as $r_t\to\infty$. Also, if $j<t$

$$\begin{aligned}\frac{N_t r_t^{n_j+h_j-n_t}}{N_j+h_j!} &\leqslant \frac{N_t^{n_j+h_j-n_t+1}}{N_j+h_j!}\\ &\leqslant \frac{N_t^{-2}}{N_j+h_j!}\\ &< \frac{1}{N_t^2},\end{aligned} \tag{5}$$

and if $j>t$

$$\begin{aligned}\frac{N_t r_t^{n_j+h_j-n_t}}{N_j+h_j!} &\leqslant \frac{N_t^{n_j+h_j-n_t+1}}{N_j+h_j!}\\ &< \frac{1}{N_j^2}.\end{aligned} \tag{5'}$$

From (4), (5) and (5′), and the fact that $f_1(z)\in M_1$, we get

$$M(r_t,f_1)\sim\mu(r_t,f_1)\sim\mu(r_t,f_2)\sim M(r_t,f_2)$$

as $r_t\to\infty$. Hence $f_2(z)\in M_1$. We have

$$f_1(z)+f_2(z)=f_3(z),$$

where

$$f_3(z)=\sum_{t=1}^{\infty}\sum_{h_t=1}^{[\log n_t]}\frac{z^{n_t+h_t}}{N_t+h_t!}.$$

If $h_j = 1$, then

$$\frac{r^{n_j+1}}{N_j+1!} \geqslant \frac{r^{n_{j-1}+1}}{N_{j-1}+1!}, \text{ i.e. from Starling's formula}$$

$$r \geqslant \left(\frac{N_j+1!}{N_{j-1}+1!}\right)(n_j - n_{j-1})^{-1} > N_j \text{ for large } j.$$

Also

$$\frac{r^{n_j+h_j}}{N_j+h_j!}\frac{N_j+1!}{r^{n_j+1}} = r^{h_j-1}\frac{N_j+1!}{N_j+h_j!} > \frac{N_j^{h_j-1}}{(N_j+2)(N_j+3)\cdots(N_j+h_j)} > \frac{1}{2}.$$

So

$$M^2(r, f_3) \geqslant \int_0^{2\pi} |f_3(re^{i\theta})|^2 \, d\theta = \sum_{t=1}^{\infty} \sum_{h_t=1}^{[\log n_t]} \frac{r^{2(n_t+h_t)}}{(N_t+h_t!)^2}$$

$$> \sum_{h_j=1}^{[\log n_j]} \frac{r^{2(n_j+h_j)}}{(N_j+h_j!)^2}$$

$$> \mu^2(r, f_3) p(r),$$

where $z = re^{i\theta}$, and $p(r) \to \infty$ as $r \to \infty$. That is,

$$\liminf_{r\to\infty} \frac{M(r, f_3)}{\mu(r, f_3)} = \infty.$$

So $f_3(z) \in M_0$. Hence the theorem is proved. □

For the proof of Theorem 3, we need the following

Lemma 1 (pp. 4–10, [1]). *If $M(r, f^{(j)})$ denotes the maximum modulus of $f^{(j)}(z)$, the jth derivative of $f(z)$, then for $j = 1, 2, \ldots$*

$$\left(\frac{\nu(r,f)}{r}\right)^j M(r,f)(1-\varepsilon_j(r)) \leqslant M(r, f^{(j)}) \leqslant \left(\frac{\nu(r,f)}{r}\right)^j,$$

where $0 < \varepsilon_j(r) \to 0$ as $r \to \infty$ outside a set S of finite logarithemic measure.

Proof of Theorem 3. Since (1) holds, there is a positive integer n_1 such that $\{|a_{n-1}/a_n|\}_{n=n_1}^{\infty}$ is a strictly increasing sequence. Let

$$f(z) + f^{(1)}(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = a_n + (n+1)a_{n+1}.$$

Let

$$a_n = |a_n| e^{i\alpha_n}, n = 0, 1, \ldots, R_n = |a_{n-1}/a_n|,$$

$$R'_n = |b_{n-1}/b_n|,$$

and

$$\alpha(n, f+f^{(1)}) = \left(\frac{1+(n-1/R_{n-1})^2 + 2(n-1/R_{n-1})\cos(\alpha_{n-2}-\alpha_{n-1})}{1+(n/R_n)^2 + 2(n/R_n)\cos(\alpha_{n-1}-\alpha_n)}\right)^{\frac{1}{2}}.$$

Then

Since $\alpha(n+1, f+f^{(1)}) \to 1$ as $n \to \infty$, there exists a positive integer n_2 such that $\{R'_n\}_{n=n_2}^{\infty}$ is a strictly increasing sequence. Let

$$E_n = [R_n, R_{n+1}] \text{ and } F_n = [R'_n, R'_{n+1}].$$

For $r \in E_n$, $n \geqslant n_1$, we have

$$\nu(r,f) = n \text{ and } \mu(r,f) = |a_n| r^n.$$

For $r \in F_n$, $n \geqslant n_2$, we have

$$\nu(r, f+f^{(1)}) = n \text{ and } \mu(r, f+f^{(1)}) = |b_n| r^n.$$

Hence for $r \in E_n$, $n \geqslant n_1$

$$\mu(r, f+f^{(1)}) \geqslant \mu(r,f)\left(1 - \frac{\nu(r,f)+1}{R_{n+1}}\right), \tag{6}$$

and for $r \in F_n$, $n \geqslant n_2$

$$\mu(r, f+f^{(1)}) \leqslant \mu(r,f)\left(1 + \frac{\nu(r, f+f^{(1)})+1}{R_{n+1}}\right). \tag{7}$$

Let $n_0 = \text{Max}\,(n_1, n_2)$. Obviously

$$\text{meas}\, \mathbb{R}_+ \setminus \bigcup_{n=n_0}^{\infty} ((E_n \setminus E_n \cap F_n) \cup (F_n \setminus E_n \cap F_n)) = \infty.$$

For

$$r \notin S_1 = S \bigcup_{n=n_0}^{\infty} ((E_n \setminus E_n \cap F_n) \cup (F_n \setminus E_n \cap F_n)),$$

we have

$$\nu(r,f) = \nu(r, f+f^{(1)}) = n.$$

So from Lemma 1, (6) and (7)

$$\frac{\mu(r,f)}{M(r,f)} \sim \frac{\mu(r, f+f^{(1)})}{M(r, f+f^{(1)})}$$

as $r \to \infty$ outside S_1. Hence the theorem is proved. □

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Linear flow induced in fluid particle suspension by an infinite differentially rotating disk

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Abstract. The steady, axisymmetric laminar flow of a homogeneous incompressible fluid with suspended particles occupying the half-infinite space over a differentially rotating rigid plane boundary is analyzed in this paper. The effect of suspended particles is described by two parameters f and τ. The mass concentration parameter f is a measure of the concentration of suspended dust particles. The interaction parameter τ is a measure of the rate at which the velocity of dust particles adjusts to changes in the fluid velocity and depends upon the size of the individual particles. Due to Ekman suction, the particle density remains no longer a constant in the boundary layer but varies with the axial coordinate ξ. Flow characteristics and density variations are studied as functions of f, τ and ξ. Possible limiting cases for $\tau \ll 1$ and $\tau \gg 1$ which correspond to the case of fine dust and coarse dust respectively are derived and discussed.

Keywords. Fluid particle suspension; rotation; linear Ekman layer.

1. Introduction

Saffman [7] who initiated the work on dusty fluids also studied the effect of dust particles on the stability of laminar flow of a dusty gas. Due to academic interest and possible applications in atmospheric, engineering and physiological fields, several authors (Michael and Miller [5], Marble [4], Datta and Mishra [1]) have studied the dynamics of dusty fluids. Zung [8] examined the flow induced in fluid-particle suspension by an infinite rotating disk. Assuming similarity solutions he obtained numerical solutions for the nonlinear problem and presented the velocity and density fields as functions of the axial distance η and the parameters of the problem. Gupta [3] studied the linear Ekman layer over a rigid rotating boundary, when the fluid motion is driven by the movement of the boundary with a uniform velocity parallel to itself. In that problem the vertical velocities of the fluid and dust particles are zero throughout and the density of the particles is a constant.

In the present paper we discuss the linear flow induced in fluid-particle suspension by an infinite differentially rotating disk. The present work differs from that of Zung [8] because this is a linear problem while that of Zung [8] is a nonlinear problem. Our

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work also differs from that of Gupta [3] for, in our problem, the axial velocities of the fluid and dust particles do not vanish, and the density of the dust particles is not a constant due to differential rotation of the disk. Due to Ekman suction the number density and hence the density of the dust particles is not a constant in the boundary layer region but is a function of the axial coordinate.

As in reference [7], in the present analysis also the effect of dust is described by two parameters f and τ which are respectively the measures of concentration of dust and the rate at which the velocity of a dust particle adjusts to changes in the fluid velocity. Solutions presented as functions of f and τ are also valid for the limiting case of $\tau \ll 1$ (fine dust) and $\tau \gg 1$ (coarse dust). Some of the qualitatively interesting results of the present analysis are (1) the boundary layer thicknesses for fluid and particle cloud are approximately equal. (2) In the case of fine dust, i.e., when $\tau \ll 1$, the velocities of the dust particles coincide with those of the fluid wherein the fluid density ρ is replaced by $\rho(1+f)$. This result is in agreement with Saffman's observation that for fine dust the particles move along streamlines with the velocity of the fluid. In this case, the density of the dusty cloud, as may be expected, is a constant. (3) In the case of coarse dust, i.e. when $\tau \gg 1$,

$$u_p = 0 + O\left(\frac{1}{\tau}\right),$$

$$v_p = 0 + O\left(\frac{1}{\tau}\right),$$

and

$$\rho_p w_p = \text{constant},$$

where ρ_p is the density and u_p, v_p, w_p are the radial, azimuthal and axial components of velocity of the dust particles. (4) The net mass flux of the fluid is perpendicular to the direction of the stress vector when $\tau \to 0$ and when $\tau \to \infty$ while it is not so for other values of τ.

2. Formulation and solution

An incompressible fluid containing small particles of a single size of radius a is occupying the half-infinite space, $z > 0$. The boundary plane at $z = 0$ and the fluid above it are rotating at a constant angular velocity Ω_0. The bounding plane is then given an extra angular velocity $\Delta\Omega$.

Introducing the non-dimensional variables

$$(u, v, w) = (u^*, v^*, w^*)/(\varepsilon\Omega_0 L),$$

$$(u_p, v_p, w_p) = (u_p^*, v_p^*, w_p^*)/(\varepsilon\Omega_0 L),$$

$$(r, z) = (r^*, z^*)/L,$$

and

$$p = p^*/(\rho\varepsilon\Omega_0^2 L^2),$$

using a cylindrical coordinate system rotating about the z-axis and taking account

(refer to [7] and [8])

$$-2v=\frac{-\partial p}{\partial r}+E\left[\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right]+\frac{f}{\tau}(u_p-u), \tag{1}$$

$$2u=E\left[\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{\partial^2 v}{\partial z^2}\right]+\frac{f}{\tau}(v_p-v), \tag{2}$$

$$0=\frac{-\partial p}{\partial z}+E\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right]+\frac{f}{\tau}(w_p-w), \tag{3}$$

$$-2v_p=\frac{1}{\tau}(u-u_p), \tag{4}$$

$$2u_p=\frac{1}{\tau}(v-v_p), \tag{5}$$

$$0=\frac{1}{\tau}(w-w_p), \tag{6}$$

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw)=0, \tag{7}$$

$$\frac{\partial}{\partial r}(r\rho_p u_p)+\frac{\partial}{\partial z}(r\rho_p w_p)=0. \tag{8}$$

Here u, v, w are components of velocity in the radial, azimuthal and axial directions and p is the reduced pressure. The non-dimensional parameters occurring in the equations are

$E=\dfrac{\nu}{\Omega_0 L^2}$, the Ekman number;

$f=\dfrac{mN_0}{\rho}$, the mass concentration parameter;

$\tau=\dfrac{m\Omega_0}{K}$, the interaction parameter,

where m and N_0 are the mass of single dust particle and the number density of the dust, and ρ, μ, ν are the density, viscosity and kinematic viscosity of the fluid respectively. Further, $K=6\pi a\mu$ for particles of radius a by Stokes' Drag formula. The subscript p denotes quantities associated with dust particles (here we are following a notation similar to that in [7]. The parameters f and τ correspond to the parameters k and $1/\beta$ of [8]).

The boundary conditions relevant to the problem are

$$u=0,\quad v=r, w=0\quad \text{for } z=0, \tag{9}$$

and

$$u=0,\quad v=0,\quad p=p_\infty,$$

Introducing the stretched coordinate $\xi(E^{-1/2}z)$ and writing any field variable F as

$$F = F^{(i)} + \bar{F},$$

where superscript i denotes the interior component and bar the boundary layer correction. The governing equations for the Ekman layer region are as follows:

$$-2\bar{v} = \frac{\partial^2 \bar{u}}{\partial \xi^2} + \frac{f}{\tau}(\bar{u}_p - \bar{u}), \tag{11}$$

$$2\bar{u} = \frac{\partial^2 \bar{v}}{\partial \xi^2} + \frac{f}{\tau}(\bar{v}_p - \bar{v}), \tag{12}$$

$$-2\bar{v}_p = \frac{1}{\tau}(\bar{u} - \bar{u}_p), \tag{13}$$

$$2\bar{u}_p = \frac{1}{\tau}(\bar{v} - \bar{v}_p), \tag{14}$$

$$0 = \frac{1}{\tau}(\bar{w} - \bar{w}_p), \tag{15}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + E^{-1/2}\frac{\partial}{\partial \xi}\bar{w} = 0, \tag{16}$$

$$\frac{\rho_p}{r}\frac{\partial}{\partial r}(r\bar{u}_p) + E^{-1/2}\frac{\partial}{\partial \xi}(\rho_p \bar{w}_p) = 0. \tag{17}$$

The pressure field is continuous and so the boundary layer correction for p to the lowest order is zero. Hence the pressure p does not appear in the equations.

Equations (11)–(16) are solved subject to relevant boundary conditions and the solutions to the lowest order are given below:

$$\bar{u} = r\exp(-T_1\xi)\cdot\sin T_2\xi, \tag{18}$$

$$\bar{v} = r\exp(-T_1\xi)\cdot\cos T_2\xi, \tag{19}$$

$$\bar{w} = \frac{2E^{1/2}}{T_1^2 + T_2^2}\exp(-T_1\xi)\cdot[T_1\sin T_2\xi + T_2\cdot\cos T_2\xi], \tag{20}$$

$$\bar{u}_p = \frac{1}{1+4\tau^2}(\bar{u} + 2\tau\bar{v}), \tag{21}$$

$$\bar{v}_p = \frac{1}{1+4\tau^2}(\bar{v} - 2\tau\bar{u}), \tag{22}$$

$$\bar{w}_p = \bar{w}, \tag{23}$$

where

$$\alpha_1 = \frac{4\tau f}{1+4\tau^2},$$

$$\alpha_2 = 2\left(1 + \frac{f}{\quad}\right)$$

$$T_1 = \{1/2[(\alpha_1^2 + \alpha_2^2)^{1/2} + \alpha_1]\}^{1/2},$$

and

$$T_2 = \{1/2[(\alpha_1^2 + \alpha_2^2)^{1/2} - \alpha_1]\}^{1/2}. \tag{24}$$

Since

$$w = w^{(i)} + \bar{w} = 0 \text{ at } \xi = 0,$$

the interior component of axial velocity is

$$w^{(i)} = -2E^{1/2}\frac{T_2}{T_1^2 + T_2^2}. \tag{25}$$

We solve (8) for the density of the dust particles to get

$$\bar{\rho}_p(\xi) = \frac{\rho_p}{f\rho} = \exp\left[\int_{\xi}^{\xi_\infty} \frac{E^{1/2}}{\bar{w}_p + w^{(i)}}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}_p) + E^{-1/2}\frac{\partial}{\partial \xi}(\bar{w}_p)\right\}\mathrm{d}\xi\right]. \tag{26}$$

In view of the relations

$$w = \bar{w} + w^{(i)} = 0 \text{ at } \xi = 0$$

and

$$\bar{w}_p = \bar{w},$$

it seems that $\bar{w}_p + w^{(i)}$ will be zero at $\xi = 0$ and hence $\xi = 0$ is a singular point of the integral in (26). However it may be noted that the equation $\bar{w}_p = \bar{w}$ is true only to the lowest order. That is, $\bar{w}_p + w^{(i)} = 0 +$ higher order terms at $\xi = 0$, where the higher order term results due to buoyancy force acting on the dust particles. In fact, the vertical velocity of the dust particles need not vanish at the boundary (see Zung [8]) and care has been taken to reflect this feature in our work.

Using the known expressions for the velocity components, the integral on the right side of relation (26) is evaluated numerically using Simpson's $-1/3$ rule to get the values of the density $\bar{\rho}_p$ at different values of ξ. We have also employed the Runge-Kutta fourth order method to integrate (8) numerically for $\bar{\rho}_p$ taking the initial condition as $\bar{\rho}_p(\xi = \xi_\infty) = 1{\cdot}0$, where ξ_∞ is taken as 10. The numerical results are found to be in good agreement with those obtained by using Simpson's rule.

When $\tau \ll 1$, from (13)–(15) we get to the lowest order

$$\bar{u}_p = \bar{u}, \quad \bar{v}_p = \bar{v}, \quad \bar{w}_p = \bar{w},$$

and from (16) and (17) it follows

$$\bar{\rho}_p = \frac{\rho_p}{f\rho} = 1,$$

thereby implying that the dust particles move along with the fluid particles. Taking the limit as $\tau \to 0$, (18)–(20) give the well-known Ekman layer solutions (Refer to pages 30–31 of Greenspan [2]) except that ρ is replaced by $\rho(1+f)$.

When $\tau \gg 1$, we rewrite (11) and (12) as (also see reference [7])

$$-2\bar{v} = \frac{\partial^2 \bar{u}}{\partial \xi^2} - s\bar{u}, \tag{27}$$

$$-2\bar{u} = \frac{\partial^2 \bar{v}}{\partial \xi^2} - s\bar{v}, \tag{28}$$

where $s = f/\tau$. Solving (27), (28) and (13)–(17) subject to appropriate boundary conditions, we get the same solutions (18)–(23) but with T_1 and T_2 defined as

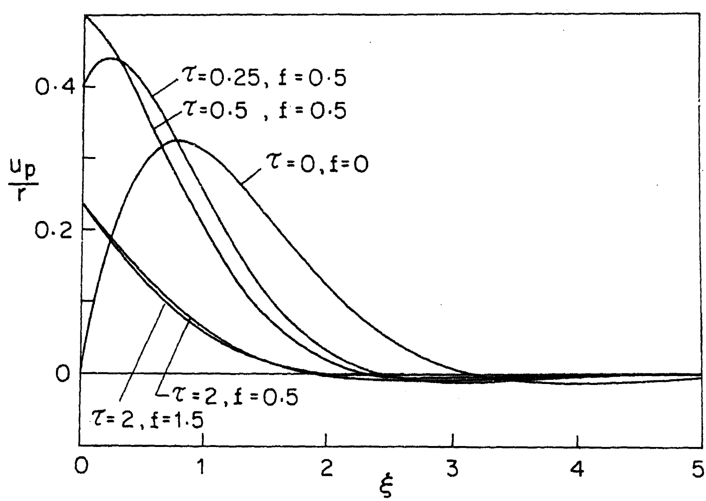
$$T_1 = \left\{ \frac{\sqrt{s^2+4}+s}{2} \right\}^{1/2},$$

$$T_2 = \left\{ \frac{\sqrt{s^2+4}-s}{2} \right\}^{1/2}.$$

These expressions for T_1 and T_2 are essentially the same as those that can be obtained from (24) in the limit of large τ, thereby establishing the fact that the effect of dust is equivalent to an extra frictional force proportional to the velocity, as pointed out in [7]. The physical explanation is that coarse dust does not move with the fluid when the flow is perturbed but carries on with the velocity of the basic flow. The disturbance has therefore to flow around the particles. Further, from the solutions (21)–(23) and equation (17) we observe $\bar{u}_p$ and $\bar{v}_p$ to be of order $1/\tau$ and the product $\rho_p \bar{w}_p$ to be a constant. These observations are in complete agreement with the numerical computations of u_p, v_p, w_p and ρ_p for large values of τ.

3. Discussion of the results

The velocity profiles of the dust particles are presented in figures 1, 2 and 3. The radial velocity (u_p) and azimuthal velocity (v_p) of the particles do not vanish on the disk. These velocities tend to zero outside the boundary layer. The axial velocity of the particles (w_p) also takes non-zero values on the disk and approach a constant value outside the boundary layer. For a fixed value of τ as f takes increasing values the magnitudes of the velocity components are found to diminish. This fact can be explained as follows. When τ is fixed and f increases, size of the particles is fixed but the number of particles N_0 increases. Then the drag force, $\mathbf{K}N_0(\mathbf{U}-\mathbf{V})$ (see Saffman [7]) increases



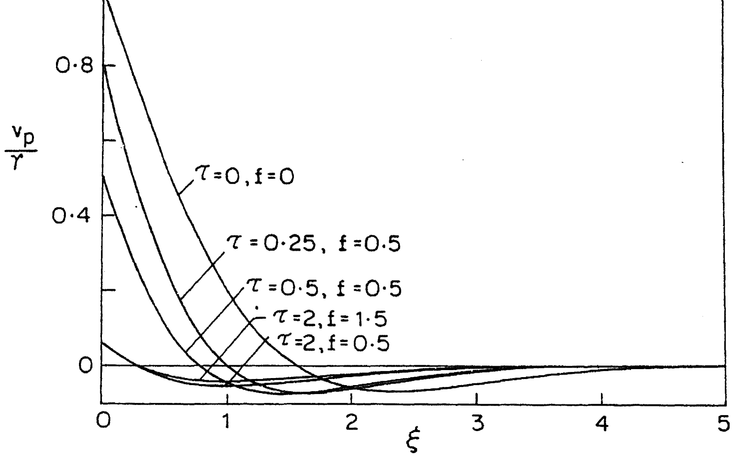


Figure 2. Dust particle velocity in azimuthal direction, v_p/r (curves 1 to 5 as in figure 1).

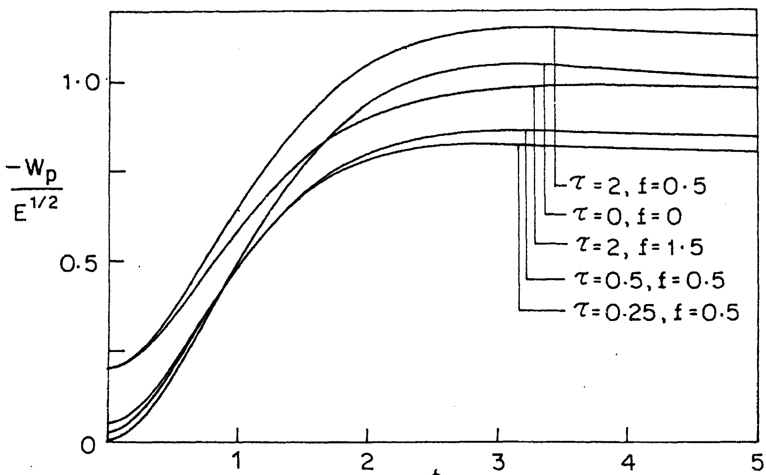


Figure 3. Dust particle velocity in axial direction, $w_p/E^{1/2}$ (curves 1 to 5 as in figure 1).

and hence the velocity decreases. Further, as f increases, the density of the suspension (fluid + dust) increases. Hence $\mu/(\rho\Omega_0 L^2)$ decreases, i.e. the Ekman layer thickness decreases. Hence w_p decreases with f. The boundary layer thickness goes like $1/(1+f)^{1/2}$.

Using (18) and (19) in (21) we get

$$\frac{\bar{u}_p}{r} = \frac{2\tau}{1+4\tau^2} \quad \text{at } \xi = 0.$$

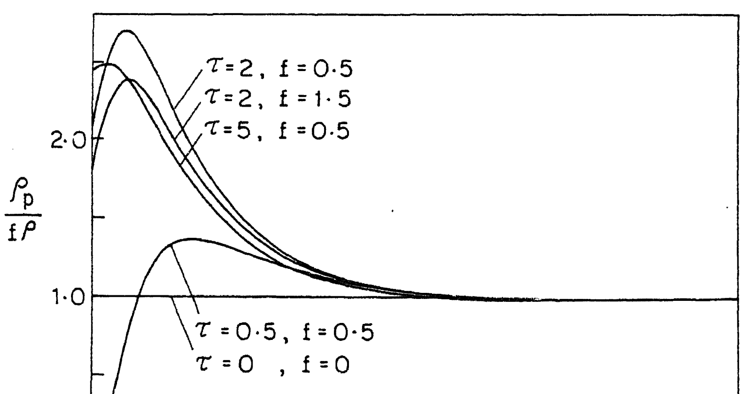
Hence we conclude that at $\xi = 0$, u_p tends to zero as $\tau \to 0$ and attains its maximum value at $\tau = 0{\cdot}5$. In fact, as $\tau \to 0$ we have $u_p = u$ as mentioned earlier; the radial flow fields for the fluid and dust particles as well have to increase from zero to a maximum

value and then decrease to zero again (see figure 1) in view of the boundary conditions. This behaviour for u_p cannot abruptly change as τ starts taking non-zero values and persists up to $\tau = 0{\cdot}5$. However the increasing portion of u_p gradually decreases as τ increases up to 0·5. This may be attributed to the fact that u_p takes increasing values at the plate with τ for $\tau < 0{\cdot}5$. For values of $\tau > 0{\cdot}5$, the particle size has increased beyond the critical value and so u_p gradually decreases at $\xi = 0$. However it will still have its maximum value at $\xi = 0$ due to centrifugal force. We may note from (4)–(6) that as $\tau \to 0$, u_p, v_p, w_p become equal to u, v, w respectively since the dust particles move with fluid. So the shapes of the curves for u_p, v_p, w_p when $\tau \to 0$ are the same as those of u, v, w of the conventional problem. The curves of particle velocities closely resemble those of the nonlinear dusty fluid problem analyzed by Zung [8] except for the fact that in the nonlinear problem u_p, v_p do not have oscillatory nature.

The plots of the particle density $\bar{\rho}_p = \rho_p/f\rho$ and the divergence of the velocity of the particles are presented in figures 4 and 5 respectively. The density $\bar{\rho}_p$ takes different values on the disk but tend to a constant value, unity, far away from the disk. The magnitude of $\bar{\rho}_p$ near the disk increases with increasing values of τ but decreases with increasing values of f. As already pointed out, for a fixed value of f, as τ increases, particle size increases and as a result, even at $\xi = 0$ the radial velocities of the particles decrease while the vertical velocities increase. Hence we expect the particle density to increase at $\xi = 0$ with τ. Further the continuity equation for the particles (8) may be rewritten as

$$\nabla \cdot \mathbf{V} = \frac{-w_p}{E^{1/2}} \frac{1}{\bar{\rho}_p} \frac{\mathrm{d}}{\mathrm{d}\xi} \bar{\rho}_p.$$

So, the sign of $\nabla \cdot \mathbf{V}$ (divergence of the velocity of the particles) depends upon the sign of $(\mathrm{d}/\mathrm{d}\xi)\bar{\rho}_p$. From figure 5 we may note that the divergence is changing its sign. Hence we can expect the density to increase and decrease. Since the divergence is positive initially up to a certain distance from the disk, particle density should increase with



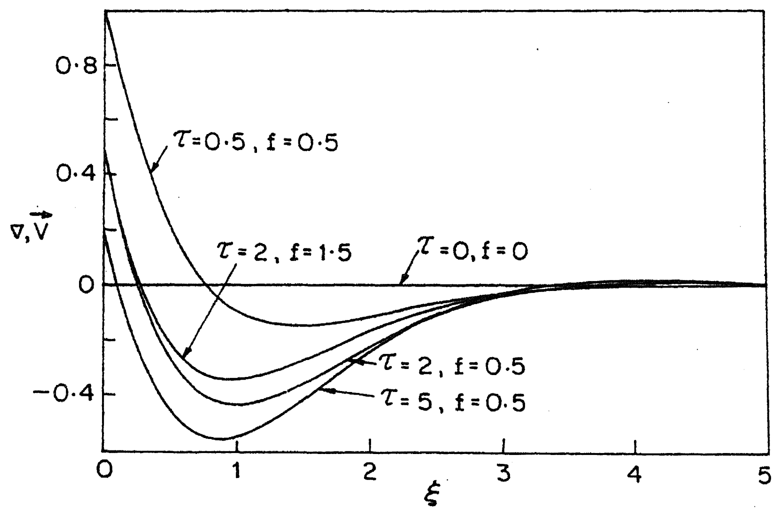


Figure 5. Divergence of dust particle velocity, $\nabla \cdot \mathbf{V}$ (curves 1 to 5 as in figure 4).

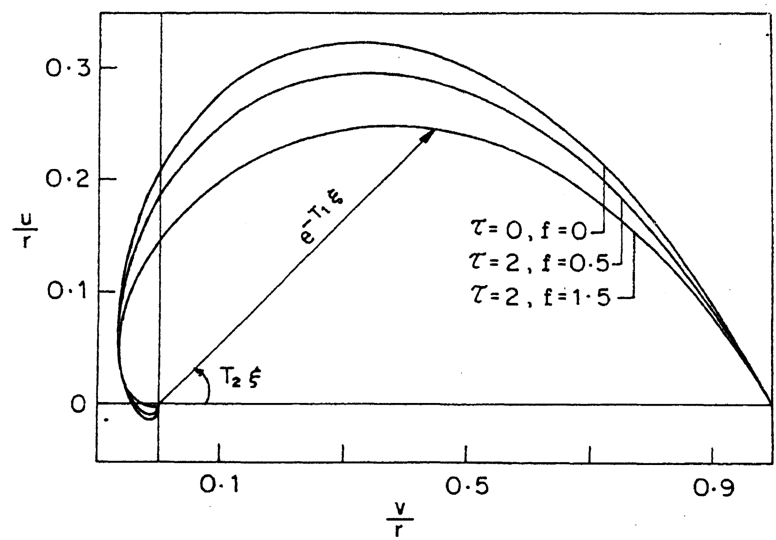


Figure 6. Ekman spiral for fluid particles.

ξ. Later, since the divergence becomes negative, $\bar{\rho}_p$ should start decreasing with ξ. Because of this behaviour $\bar{\rho}_p$ can attain values greater than unity in the boundary layer!

The velocity vectors of the fluid and dust particles at different heights above the rigid boundary are shown in figures 6 and 7. As ξ increases the velocity vector rotates

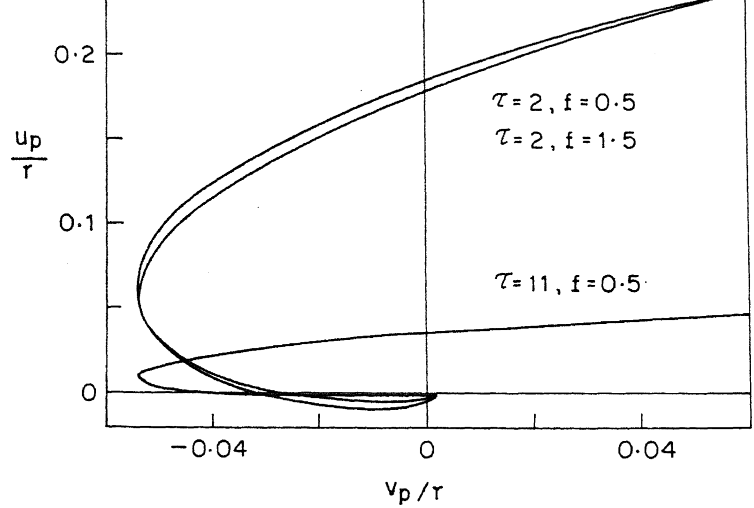


Figure 7. Ekman spiral for dust particles.

4. Net mass flux

The net mass flux of the fluid can be obtained by integrating the velocity vector from $z=0$ to $z=\delta$ (δ is the boundary layer thickness) as

$$\begin{aligned}\mathbf{M} &= \int_0^\delta \mathbf{U}\,\mathrm{d}z \\ &= E^{1/2}\int_0^\infty \mathbf{U}\,\mathrm{d}\xi \\ &= \frac{rE^{1/2}}{T_1^2+T_2^2}(T_2\hat{r}+T_1\hat{\theta}).\end{aligned} \tag{29}$$

The stress vector at the disk can be obtained to be

$$\mathbf{T} = rE^{1/2}(-T_2\hat{r}+T_1\hat{\theta}). \tag{30}$$

From (29) and (30) it follows that

$$\mathbf{M}\cdot\mathbf{T} = \frac{r^2}{T_1^2+T_2^2}(-T_2^2+T_1^2). \tag{31}$$

So, in the presence of dust the net mass flux is not perpendicular to the direction of the stress vector. However, for fine dust, in the limit as $\tau\to 0$, we get from (31)

$$\mathbf{M}\cdot\mathbf{T} = 0,$$

implying that the net mass flux is perpendicular and to the right of the stress vector. This is in agreement with the well established fact that for a conventional dust free fluid the total mass flow in excess of the prescribed geostrophic flow is perpendicular

[6]). For coarse dust, when $\tau \gg 1$, from (31) we have

$$\mathbf{M}\cdot\mathbf{T} = \frac{r^2 s}{(s^2+4)^{1/2}},$$

where

$$s = \frac{f}{\tau}.$$

5. Turning moment

The results obtained and the calculations done are applicable to an infinite disk only. However, we may utilize the results for a finite disk provided that its radius R is large compared to the boundary layer thickness. The turning moment for a disk of radius R is

$$M_t = -2\pi \int_0^R r^2 T_{z\theta}\, \mathrm{d}r,$$

where the radial distance r is in dimensional form and the circumferential component of the shearing stress $T_{z\theta}$ is

$$T_{z\theta} = \mu\left(\frac{\partial v}{\partial z}\right)_0 = \varepsilon\rho r \nu^{1/2}\Omega_0^{3/2}\cdot T_1.$$

Here T_1 is as defined in (24). The coefficient of turning moment (i.e. the dimensionless moment coefficient) is

$$C_M = \frac{M_t}{\frac{1}{2}\rho\Omega_0^2 R^5} = \frac{\varepsilon\pi T_1}{\sqrt{Re}}.$$

Here Re $(= R^2\Omega_0/\nu)$ is the Reynolds number based on the radius and tip velocity.

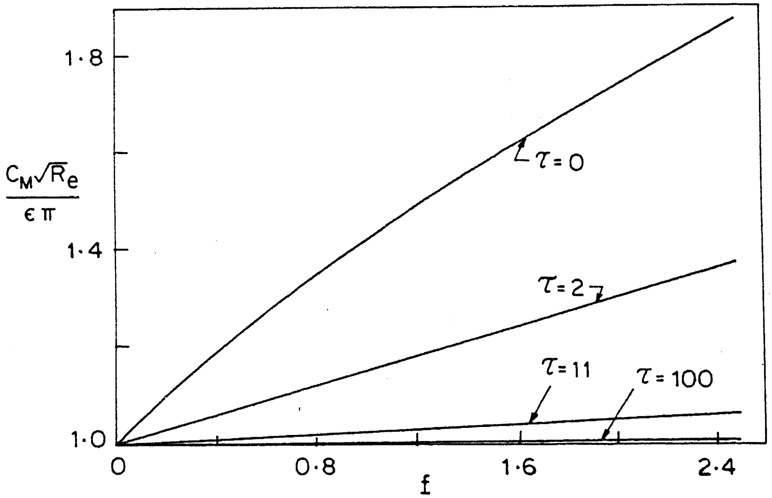


Figure 8. Coefficient of turning moment, $C_M\sqrt{Re}/\varepsilon\pi$.

Figure 8 shows the plots of the coefficient of turning moment, $C_M\sqrt{Re}/\varepsilon\pi$. C_M increases with increasing values of f and this is in agreement with the fact that an increase in f causes a decrease in boundary layer thickness. C_M decreases with increasing values of τ.

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Symbols

C_M	:	coefficient of turning moment
E	:	Ekman number
$\mathbf{M}$	:	mass flux
M_t	:	turning moment
N_0	:	number density of dust particles
$\mathbf{U}=(u, v, w)$	:	non-dimensional velocity of fluid
$\mathbf{V}=(u_p, v_p, w_p)$	:	non-dimensional velocity of the dust particles
Z	:	axial coordinate
f	:	mass concentration parameter
p	:	reduced pressure
r	:	radial coordinate
ε	:	Rossby number
τ	:	interaction parameter
Ω_0	:	angular velocity
ρ_p	:	density of dust particles in the suspension.

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Integrals involving Fox's H-function

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Abstract. We evaluate four integrals involving Fox's H-functions and a general class of polynomials $S_n^m[x]$, introduced earlier by Srivastava.

Keywords. H-function; generalized polynomials; extended Jacobi polynomials.

1. Introduction and definitions

Recently Kalla *et al* [1] and Kalla [2] established a number of integrals involving Jacobi polynomials and generalized Jacobi functions. The aim of this paper is to evaluate four integrals involving Fox's H-function and a general class of polynomials $S_n^m[x]$. The technique followed is essentially that of Kalla [1], [2]. Srivastava [4] studied the general class of polynomials $S_n^m[x]$, defined as

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots \tag{1}$$

where m is an arbitrary positive integer, the coefficients $A_{n,k}(n, k \geqslant 0)$ are arbitrary constants, real or complex, and

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}. \tag{2}$$

By suitably specialising the coefficients $A_{n,k}$, the polynomials $S_n^m[x]$ can be reduced to the classical orthogonal polynomials (see Srivastava and Singh [5] for details). The Fox's H-function is defined and represented as follows [6]:

$$\begin{aligned} H(x) &= H_{P,Q}^{M,N}\left[x \left| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right.\right] \\ &= \frac{1}{2\pi i} \int_{\mathscr{L}} \theta(s) x^s \, \mathrm{d}s, \end{aligned} \tag{3}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j s) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s)}. \tag{4}$$

By summing up the residues at the simple poles of the integrand of (3), the following expression for $H[x]$ was derived by Braaksma [see 6]:

$$H[x] = \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left\{ \frac{(-1)^r}{r!} \frac{\phi(\lambda)}{\beta_h} x^{\lambda} \right\}, \tag{5}$$

where

$$\lambda = \frac{b_h + r}{\beta_h}, \quad r = 0, 1, 2, \ldots \tag{6}$$

and

$$\phi(t) = \frac{\theta(t)}{\Gamma(b_h - \beta_h t)}, \tag{7}$$

provided that the series on the right side of (5) is absolutely convergent.

2. Result required

The following integral is required to establish the main integrals:

$$\int_{-1}^{1} (1-x)^a (1+x)^b S_n^m \left[c \left(\frac{1-x}{2}\right)^{\delta_1} \left(\frac{1+x}{2}\right)^{\delta_2} \right]$$

$$\times H_{P,Q}^{M,N} \left[z \left(\frac{1-x}{2}\right)^{\mu} \left(\frac{1+x}{2}\right)^{\nu} \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] \mathrm{d}x$$

$$= 2^{a+b+1} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} H_{P+2,Q+1}^{M,N+2} \left[z \middle| \begin{matrix} (-a-\delta_1 k, \mu), \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right.,$$

$$\left. \begin{matrix} (-b-\delta_2 k, \nu), (a_j, \alpha_j)_{1,P} \\ (-1-a-b-\delta_1 k - \delta_2 k, \mu+\nu) \end{matrix} \right] c^k, \tag{8}$$

$$= 2^{a+b+1} \sum_{k=0}^{[n/m]} \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left\{ \frac{(-n)_{mk}}{k!} \frac{(-1)^r}{r!} f(\lambda) A_{n,k} c^k \frac{z^{\lambda}}{\beta_h} \right\} \tag{9}$$

with

$$\lambda = \frac{b_h + r}{\beta_h}, \quad r = 0, 1, 2, \ldots$$

and

$$f(t) = \phi(t) B(1 + a + \delta_1 k + \mu t, 1 + b + \delta_2 k + \nu t), \tag{10}$$

provided that the following conditions are satisfied:

(i) $A > 0$, $\delta > 0$, $|\arg(z)| < (1/2) A\pi$
where

$$A = \sum_{j=1}^{N} (\alpha_j) - \sum_{j=N+1}^{P} (\alpha_j) + \sum_{j=1}^{M} (\beta_j) - \sum_{j=M+1}^{Q} (\beta_j) \tag{11}$$

(iii) $\quad \mu \min\limits_{1\leqslant j\leqslant M} [\mathrm{Re}(b_j/\beta_j)]+1>0,$

$$\nu \min_{1\leqslant j\leqslant M} [\mathrm{Re}(b_j/\beta_j)]+1>0.$$

The result in (8) is easily established when we replace the H-function by its Mellin-Barnes contour integral from (3), interchange the order of integrations (which is justified due to absolute convergence of the integrals involved in the process), replace $S_n^m[x]$ by its series representation with the help of (1) and then integrate term by term with the help of the result [3], viz.

$$\int_{-1}^{1} (1-x)^{a-1}(1+x)^{b-1}\,\mathrm{d}x = 2^{a+b-1}B(a,b), \mathrm{Re}(a)>0, \mathrm{Re}(b)>0. \tag{13}$$

(See also Theorem 1 of [5] for a multivariable H-function integral analogous to (8) above.)

3. Main integrals

If $\psi(z)$ denotes the logarithmic derivative of the gamma function $\Gamma(z)$, i.e., $\psi(z)=\Gamma'(z)/\Gamma(z)$, then with $\lambda=(b_h+r)/(\beta_h)$, $r=0,1,2,\ldots$ and $f(t)$ given by (10), we have, for $A>0$, $\delta>0$, $|\arg z|<1/2\,A\pi$(A and δ being given by (11) and (12), $a,b,\delta_1,\delta_2,\mu,\nu>0$ and

$$\mu \min_{1\leqslant j\leqslant M} [\mathrm{Re}(b_j/\beta_j)]+1>0, \quad \nu \min_{1\leqslant j\leqslant M} [\mathrm{Re}(b_j/\beta_j)]+1>0$$

and

$$F(x)=(1-x)^a(1+x)^b S_n^m\left[c\left(\frac{1-x}{2}\right)^{\delta_1}\left(\frac{1+x}{2}\right)^{\delta_2}\right] \times H_{P,Q}^{M,N}\left[z\left(\frac{1-x}{2}\right)^{\mu}\left(\frac{1+x}{2}\right)^{\nu}\right], \tag{14}$$

$$\int_{-1}^{1} F(x)\log(1-x)\,\mathrm{d}x = 2^{b+1}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}}{k!}A_{n,k}c^k\frac{\partial}{\partial a}\left\{2^a H_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}(-a-\delta_1 k,\mu),(-b-\delta_2 k,\nu),(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q},(-1-a-b-\delta_1 k-\delta_2 k,\mu+\nu)\end{matrix}\right.\right]\right\}\right],$$

$$= 2^{a+b+1}\sum_{k=0}^{[n/m]}\sum_{h=1}^{M}\sum_{r=0}^{\infty}\left[\frac{(-n)_{mk}}{k!}\frac{(-1)^r}{r!}\frac{f(\lambda)}{\beta_h}A_{n,k}\{\log 2+\psi(1+a+\delta_1 k+\mu\lambda) - \psi(2+a+b+(\delta_1+\delta_2)k+(\mu+\nu)\lambda)\}c^k z^\lambda\right]; \tag{15}$$

$$\int_{-1}^{1} F(x)\log(1+x)\,\mathrm{d}x = 2^{a+1}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}}{k!}A_{n,k}c^k\frac{\partial}{\partial b}\left\{2^b H_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}(-a-\delta_1 k,\mu),(-b-\delta_2 k,\nu),(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q},(-1-a-b-\delta_1 k-\delta_2 k,\mu+\nu)\end{matrix}\right.\right]\right\}\right]$$

$$= 2^{a+b+1} \sum_{k=0}^{[n/m]} \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left[\frac{(-n)_{mk}}{k!} \frac{(-1)^r}{r!} \frac{f(\lambda)}{\beta_h} A_{n,k} \{\log 2 + \psi(1+b+\delta_2 k + \nu\lambda) - \psi(a+b+2+(\delta_1+\delta_2)k+(\mu+\nu)\lambda)\} c^k z^\lambda \right]; \tag{16}$$

$$\int_{-1}^{1} F(x) \log(1-x^2)\, dx = 2^{a+b+1} \sum_{k=0}^{[n/m]} \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left[\frac{(-n)_{mk}}{k!} \frac{(-1)^r}{r!} \frac{f(\lambda)}{\beta_h} A_{n,k} \times \{\log 4 + \psi(1+a+\delta_1 k + \mu\lambda) + \psi(1+b+\delta_2 k + \nu\lambda) - 2\psi(a+b+2+(\delta_1+\delta_2)k+(\mu+\nu)\lambda)\} c^k z^\lambda \right]; \tag{17}$$

$$\int_{-1}^{1} F(x) \log\left(\frac{1-x}{1+x}\right) dx = 2^{a+b+1} \sum_{k=0}^{[n/m]} \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left[\frac{(-n)_{mk}}{k!} \frac{(-1)^r}{r!} \frac{f(\lambda)}{\beta_h} A_{n,k} \times \{\psi(1+a+\delta_1 k + \mu\lambda) - \psi(1+b+\delta_2 k + \nu\lambda)\} c^k z^\lambda \right]. \tag{18}$$

Outline of Proof: The results in (15) and (16) are established by taking the partial derivatives of both the sides of (9) with respect to a and b respectively. The integrals in (17) and (18) are obtained by first adding (15) and (16) and then subtracting them.

4. Particular cases

1. If, in (17) we take

$$A_{n,k} = \frac{(\gamma+n)_k \prod_{j=1}^{p} (\varepsilon_j)_k}{\prod_{j=1}^{q} (\eta_j)_k} = D_{n,k}, \text{ say} \tag{19}$$

and $m = 1$, it reduces to the following integral involving extended Jacobi polynomials [5]:

With
$$G(x) = (1-x)^a (1+x)^b H_{P,Q}^{M,N}\left[z\left(\frac{1-x}{2}\right)^\mu \left(\frac{1+x}{2}\right)^\nu \right], \tag{20}$$

$$\int_{-1}^{1} G(x) \log(1-x^2)\, {}_{p+2}F_q\left[\begin{matrix} -n, \gamma+n, & (\varepsilon_p) \\ & (\eta_q) \end{matrix}; c\left(\frac{1-x}{2}\right)^{\delta_1} \left(\frac{1+x}{2}\right)^{\delta_2} \right] dx$$

$$= 2^{a+b+1} \sum_{k=0}^{n} \frac{(-n)_k}{k!} D_{n,k} c^k E_1(M, a, b, k), \tag{21}$$

where

$$E_1(M, a, b, k) = \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left\{ \frac{(-1)^r}{r!} \frac{f(\lambda)}{\beta_h} L_1(a, b, k, r) z^\lambda \right\} \tag{22}$$

with

$$L_1(a, b, k, r) = \{\log 4 + \psi(1+a+\delta_1 k + \mu\lambda) + \psi(1+b+\delta_2 k + \nu\lambda) - 2\psi(a+b+2+(\delta_1+\delta_2)k+(\mu+\nu)\lambda)\},$$

$\lambda = (b_h + r)/\beta_h$, $r = 0, 1, 2, \ldots$ and $f(\lambda)$ is given by (10); provided that the conditions (i), (ii) and (iii) given with the integral (8) are satisfied.

2. If, in (18), we take

$$A_{n,k} = \frac{\prod_{j=1}^{l-1}\left(\frac{\gamma+n+j}{l}\right)_k \prod_{j=1}^{p}(\varepsilon_j)_k}{\prod_{j=1}^{q}(\eta_j)_k} = B_{n,k}, \quad \text{say} \tag{23}$$

and $c = m^{-m}$ so that $S_n^m[x]$ reduces to the generalized extended Jacobi polynomial [5].

$${}_{l+m+p}F_q\left[\begin{matrix}\Delta(m;-n), \Delta(l;\gamma+n), & (\varepsilon_p) \\ & (\eta_q)\end{matrix}; x\right],$$

then (18) gives the following integral:

With $G(x)$ given by (20), we have

$$\int_{-1}^{1} G(x)\log\left(\frac{1-x}{1+x}\right) {}_{l+m+p}F_q\left[\begin{matrix}\Delta(m;-n), \Delta(l;\gamma+n), & (\varepsilon_p) \\ & (\eta_q)\end{matrix};\right.$$

$$\left.\times\left(\frac{1-x}{2}\right)^{\delta_1}\left(\frac{1+x}{2}\right)^{\delta_2}\right]\mathrm{d}x$$

$$= 2^{a+b+1}(n)! \sum_{k=0}^{[n/m]} \frac{(-1/m)^{mk}}{k!(n-mk)!} B_{n,k} E_2(M,a,b,k) \tag{24}$$

where

$$E_2(M,a,b,k) = \sum_{h=1}^{M} \sum_{r=0}^{\infty} \left\{\frac{(-1)^r}{r!}\frac{f(\lambda)}{\beta_h} L_2(a,b,k,r) z^{\lambda}\right\} \tag{25}$$

with $L_2(a,b,k,r) = \psi(1 + a + \delta_1 k + \mu\lambda) - \psi(1 + b + \delta_2 k + \nu\lambda)$, $\lambda = b_h + r/\beta_h$, $r = 0,1,2\ldots$ and $f(\lambda)$ is given by (10); provided that the conditions (i), (ii) and (iii) given with the integral (8) are satisfied.

A number of other particular cases can be obtained from the main integrals but these are not recorded here for lack of space.

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An extension of bilateral generating functions of modified Laguerre polynomials

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Abstract. In this note a theorem concerning the extension of bilateral generating functions of the modified Laguerre polynomials is derived. Some applications of the theorem are also pointed out.

Keywords. Modified Laguerre polynomials; bilateral generating functions.

1. Introduction

Recently the present authors [5] have studied the following theorem on bilateral generating functions of modified Laguerre polynomials, defined by Goyal [3]:

If there exists an unilateral generating relation of the form

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_{a,b,m,n}(x) w^n \tag{1}$$

then we have

$$(1 - wb)^{-m} \exp\left(\frac{-wax}{1 - wb}\right) G\left(\frac{x}{1 - wb}, \frac{wz}{1 - wb}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(z) L_{a,b,m,n}(x) \cdot w^n \tag{2}$$

where

$$\sigma_n(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k. \tag{3}$$

The aim of this note is to state and prove a theorem in connection with the extension of bilateral generating functions for the modified Laguerre polynomials. In fact, the following theorem is obtained as the main result of our investigation:

where k is a non-negative integer, then we have

$$(1-wb)^{-m-k}\exp\left(\frac{-wax}{1-wb}\right)G\left(\frac{x}{1-wb},\frac{wz}{1-wb}\right)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\,L_{a,b,m,n+k}(x)\cdot w^n \tag{5}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n}a_p\binom{n+k}{p+k}z^p. \tag{6}$$

The importance of the above theorem lies in the fact that a large number of generating relations can be obtained from (5) by attributing different suitable values to a_n in the relation (4).

Some interesting applications and special cases have also been shown.

Proof of the theorem. We shall first consider the following linear partial differential operator

$$R=bxy\frac{\partial}{\partial x}+by^2\frac{\partial}{\partial y}+y\{b(m+k)-ax\} \tag{7}$$

such that

$$R[L_{a,b,m,n+k}(x)y^n]=(n+k+1)\cdot L_{a,b,m,n+k+1}(x)\cdot y^{n+1}. \tag{8}$$

The extended form of the group generated by R is given by

$$\exp(wR)[f(x,y)]=(1-wby)^{-m-k}\exp\left(\frac{-waxy}{1-wby}\right)f\left(\frac{x}{1-wby},\frac{y}{1-wby}\right) \tag{9}$$

Consider the following formula

$$G(x,w)=\sum_{n=0}^{\infty}a_n\cdot L_{a,b,m,n+k}(x)\cdot w^n. \tag{10}$$

Replacing w by wyz in (10) we get

$$G(x,wyz)=\sum_{n=0}^{\infty}a_n\{L_{a,b,m,n+k}(x)\cdot y^n\}(wz)^n. \tag{11}$$

Operating $\exp(wR)$ on both sides of (11) we get

$$\exp(wR)\cdot G(x,wyz)=\exp(wR)\left[\sum_{n=0}^{\infty}a_n\{L_{a,b,m,n+k}(x)\cdot y^n\}(wz)^n\right]. \tag{12}$$

The left member of (12) on recalling (9) becomes

$$(1-wby)^{-m-k}\exp\left(\frac{-waxy}{1-wby}\right)G\left(\frac{x}{1-wby},\frac{wyz}{1-wby}\right). \tag{13}$$

Also the right member of (12) with the help of (8) may be written as

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\cdot(wz)^n\cdot\frac{w^p}{p!}\cdot R^p\{L_{a,b,m,n+k}(x)y^n\}$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n\cdot\frac{(wy)^{n+p}}{p!}\cdot z^n\cdot(n+k+1)_p\cdot L_{a,b,m,n+k+p}(x)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p}\cdot\frac{(wy)^n}{p!}\cdot(n-p+k+1)_p\cdot z^{n-p}\cdot L_{a,b,m,n+k}(x)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{p=0}^{n} a_{n-p}\cdot\frac{(n-p+k+1)_p}{p!}\cdot z^{n-p}\right)(wy)^n\cdot L_{a,b,m,n+k}(x)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{a,b,m,n+k}(x)\cdot(wy)^n$$

where

$$\sigma_n(z)=\sum_{p=0}^{n} a_p\cdot\binom{n+k}{p+k}\cdot z^p. \tag{14}$$

Equating (13) and (14) and then putting $y=1$ we get

$$(1-wb)^{-m-k}\cdot\exp\left(\frac{-wax}{1-wb}\right)\cdot G\left(\frac{x}{1-wb},\frac{wz}{1-wb}\right)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{a,b,m,n+k}(x)\cdot w^n$$

where

$$\sigma_n(z)=\sum_{p=0}^{n} a_p\binom{n+k}{p+k}\cdot z^p. \tag{15}$$

This completes the proof of the theorem.

Putting $k=0$, we get the following result on bilateral generating relation as a corollary to our main theorem:

COROLLARY 1

If there exists an unilateral generating relation of the form

$$G(x,w)=\sum_{n=0}^{\infty} a_n\cdot L_{a,b,m,n}(x)\cdot w^n \tag{16}$$

then, we have

$$(1-wb)^{-m}\cdot\exp\left(\frac{-wax}{1-wb}\right)\cdot G\left(\frac{x}{1-wb},\frac{wz}{1-wb}\right)$$

$$=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{a,b,m,n}(x)\cdot w^n \tag{17}$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \cdot \binom{n}{p} \cdot z^p \tag{18}$$

which is found derived in [5].

2. Applications

As an application we consider the following generating relation [6]:

$$\sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} \cdot L_{a,b,m,n+k}(x) \cdot w^n$$
$$= (1-wb)^{-m-k} \cdot \exp\left(\frac{-axy}{1-by}\right) \cdot L_{a,b,m,k}\left(\frac{x}{1-wb}\right). \tag{19}$$

Putting $a_n = (k+1)_n/n!$ in our theorem we get the following generalization of (19),

$$(1-wb-wbz)^{-m-k} \cdot \exp\left[\frac{-wax(1+z)}{1-wb-wbz}\right]$$
$$= \sum_{n=0}^{\infty} \sigma_n(z) \cdot L_{a,b,m,n+k}(x) \cdot w^n \tag{20}$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} \binom{p+k}{p} \cdot \binom{n+k}{p+k} \cdot z^p. \tag{21}$$

3. Special cases

(a) On specializing the parameters as $a = b = 1$ and $m = (1+\alpha)$ in our main theorem we get the following results on Laguerre polynomials:

Theorem. *If there exists an unilateral generating relation of the form*

$$G(x, w) = \sum_{n=0}^{\infty} a_n \cdot L_{n+k}^{(\alpha)}(x) \cdot w^n \tag{22}$$

where k is a non-negative integer, then we have

$$(1-w)^{-1-\alpha-k} \cdot \exp\left(\frac{-wx}{1-w}\right) \cdot G\left(\frac{x}{1-w}, \frac{wz}{1-w}\right)$$
$$= \sum_{n=0}^{\infty} \sigma_n(z) \cdot L_{n+k}^{(\alpha)}(x) \cdot w^n \tag{23}$$

where

COROLLARY 2

If there exists an unilateral generating relation of the form

$$G(x,w)=\sum_{n=0}^{\infty} a_n\cdot L_n^{(\alpha)}(x)\cdot w^n \tag{25}$$

then we have

$$(1-w)^{-1-\alpha}\cdot\exp\left(\frac{-wx}{1-w}\right)\cdot G\left(\frac{x}{1-w},\frac{wz}{1-w}\right)=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_n^{(\alpha)}(x)\,w^n. \tag{26}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n} a_p\cdot\binom{n}{p}\cdot z^p \tag{27}$$

which is found derived by [1] *and* [2].

(b) Again assigning $a=b=1$ and $m=(1+\alpha)$ in (20) we get

$$(1-w-wz)^{-1-\alpha-k}\cdot\exp\left[\frac{-wx(1+z)}{1-w-wz}\right]=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_{n+k}^{(\alpha)}(x)\cdot w^n \tag{28}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n}\binom{p+k}{p}\cdot\binom{n+k}{p+k}\cdot z^p \tag{29}$$

Putting $k=0$ in (25) and (26) we get the following result:

$$(1-w-wz)^{-1-\alpha}\exp\left[\frac{-wx(1+z)}{1-w-wz}\right]=\sum_{n=0}^{\infty}\sigma_n(z)\cdot L_n^{(\alpha)}(x)\cdot w^n \tag{30}$$

where

$$\sigma_n(z)=\sum_{p=0}^{n}\binom{n}{p}\cdot z^p \tag{31}$$

which is the result obtained in [5].

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Bilateral generating functions for Jacobi polynomials

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Abstract. In this paper, we have obtained three theorems on generating functions. We derive from these theorems a large number of bilateral generating functions for Jacobi polynomials. Certain interesting expansions of triple hypergeometric series are also obtained from one of the theorems.

Keywords. Bilateral generating functions; orthogonal polynomials; generating relations; Jacobian polynomials; triple hypergeometric series.

1. Introduction

With the usual notation, the general triple hypergeometric series is defined by Srivastava [7] as follows:

$$F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\,x,y,z\right] \tag{1}$$

$$= \sum_{r,s,t=0}^{\infty} \frac{[(a)]_{r+s+t}[(b)]_{r+s}[(b')]_{s+t}[(b'')]_{r+t}}{[(e)]_{r+s+t}[(g)]_{r+s}[(g')]_{s+t}[(g'')]_{r+t}} \times \frac{[(c)]_r[(c')]_s[(c'')]_t}{[(h)]_r[(h')]_s[(h'')]_t} \frac{x^r y^s z^t}{r!\,s!\,t!}$$

where

$$(\alpha)n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\begin{cases}1, & \text{if } n=0,\\ (\alpha)(\alpha+1)\dots(\alpha+n-1), & \forall\, n\in\{1,2,3,\dots\},\end{cases} \tag{2}$$

and, for the sake of brevity, (a) is taken to abbreviate the sequence of A parameters $a_1, a_2, \dots, a_A$ and so on, and

$$[(a)]_n = \prod_{n=0}^{A} (a_j)n, \text{etc.}$$

Theorem 1. *Let $\{A_n\}$ be a sequence of arbitrary complex numbers, then we have*

$$= \sum_{r,s,t=0}^{\infty} \frac{[(a)]_{r+s+t}[(b)]_{r+s}[(b')]_{s+t}[(b'')]_{r+t}[(c)]_r[(c')]_s}{[(e)]_{r+s+t}[(g)]_{r+s}[(g')]_{s+t}[(g'')]_{r+t}[(h)]_r[(h')]_s}$$

$$\times \frac{[(c'')]_t}{[(h'')]_t} \frac{x^r y^s z^t}{r!s!t!} \sum_{n=0}^{r} A_n \frac{(-r)_n \Gamma(1+\delta+2n)}{\Gamma(1+\delta+n+r)} x^{-n}(-1)^n. \tag{3}$$

Proof. Denoting the first member of (3) by Ω and using the definition (1), it is readily observed that

$$\Omega = \sum_{n,r,s,t=0}^{\infty} \frac{[(a)]_{r+s+t+n}[(b)]_{r+s+n}[(b')]_{s+t}[(b'')]_{r+t+n}}{[(e)]_{r+s+t+n}[(g)]_{r+s+n}[(g')]_{s+t}[(g'')]_{r+t+n}}$$

$$\times \frac{[(c)]_{r+n}[(c')]_s[(c'')]_t}{[(h)]_{r+n}(1+\delta+2n)_r[(h')]_s[(h'')]_t} A_n \frac{x^r y^s z^t}{r!s!t!}$$

$$= \sum_{r,s,t=0}^{\infty} \sum_{n=0}^{r} \frac{[(a)]_{r+s+t}[(b)]_{r+s}[(b')]_{s+t}[(b'')]_{r+t}}{[(e)]_{r+s+t}[(g)]_{r+s}[(g')]_{s+t}[(g'')]_{r+t}}$$

$$\times \frac{[(c)]_r[(c')]_s[(c'')]_t}{[(h)]_r(1+\delta+2n)_{r-n}[(h')]_s[(h'')]_t} A_n \frac{x^{r-n} y^s z^t}{(r-n)!s!t!}$$

which, in view of (2), is precisely the second member of (3). This evidently completes the proof of Theorem 1 under the assumption that the interchange of the order of summations is permissible by the absolute convergence of the series involved. Now we recall the familiar expansion ([3], p. 212)

$$[\tfrac{1}{2}(1-w)]^r = \Gamma(r+\alpha+1)$$

$$\times \sum_{n=0}^{r} \frac{(-r)n(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+r+\alpha+\beta+2)} P_n^{(\alpha,\beta)}(w), \tag{4}$$

where $P_n^{(\alpha,\beta)}(w)$ is the classical Jacobi polynomial ([3], p. 170). Now replacing δ by $1+\alpha+\beta$, A_n by $(-x)^n/(1+\alpha+\beta+n)_n(1+\alpha)_n P_n^{(\alpha,\beta)}(w)$ and using the expansion (4) we obtain the following bilateral generating function from (3):

$$\sum_{n=0}^{\infty} \frac{[(a)]_n[(b)]_n[(b'')]_n[(c)]_n(-x)^n P_n^{(\alpha,\beta)}(w)}{[(e)]_n[(g)]_n[(g'')]_n[(h)]_n(1+\alpha)_n(1+\alpha+\beta+n)_n}$$

$$\times F^{(3)}\left[\begin{array}{l} (a)+n::(b)+n;(b');(b'')+n:(c)+n;(c');(c''); \\ (e)+n::(g)+n;(g');(g'')+n:(h)+n, \\ \quad 2+\alpha+\beta+2n;(h');(h''); \end{array} x,y,z\right]$$

$$= F^{(3)}\left[\begin{array}{l} (a)::(b);(b');(b''):(c);(c');(c''); \\ (e)::(g);(g');(g''):(h),1+\alpha;(h');(h''); \end{array} \tfrac{1}{2}x(1-w),y,z\right]. \tag{5}$$

The above result (5) may also be obtained from the special case $r=3$ of Theorem 3 of [9].

Application. Since the triple hypergeometric series $F^{(3)}$ is a unification of Lauricella's functions $F_A, F_B, \ldots, F_T$ ([5], p. 114) (see also [11]) and Srivastava's functions H_A,

functions are deducible from the result (5). Thus, for example, if we set in (5) $A = C = C' = C'' = H' = H'' = 1$, $B = B' = B'' = E = G = G' = G'' = H = 0$, we shall obtain the result (3.7) of [9]. Similarly on specializing the parameters, the result (3.8) of [9] and other results involving these functions follow from (5). For more general multivariable results on generating functions of this type and other, see the works of Srivastava and Pathan ([9] and [10]; see also [12]).

Expansions. Next, setting

$$A_n = \frac{(\delta)_n(\alpha)_n(\beta)_n}{(\frac{1}{2}\delta)_n[\frac{1}{2}(1+\delta)]_n(1+\delta-\alpha)_n(1+\delta-\beta)_n}(-\tfrac{1}{4}x)^n \tag{6}$$

and recalling the following sum ([2], p. 25)

$${}_5F_4\left[\begin{matrix} -r, \delta, 1+\frac{1}{2}\delta, \alpha, \beta; \\ \frac{1}{2}\delta, 1+\delta-\alpha, 1+\delta-\beta, 1+\delta+r; \end{matrix} 1\right] = \frac{(1+\delta)_r(1+\delta-\alpha-\beta)_r}{(1+\delta-\alpha)_r(1+\delta-\beta)_r} \tag{7}$$

we obtain from Theorem 1:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[(a)]_n[(b)]_n[(b'')]_n[(c)]_n(\alpha)_n(\beta)_n(-x)^n}{[(e)]_n[(g)]_n[(g'')]_n[(h)]_n(\delta+n)_n(1+\delta-\alpha)_n(1+\delta-\beta)_n} \\
&\quad \times F^{(3)}\left[\begin{matrix} (a)+n::(b)+n;(b');(b'')+n:(c)+n;(c');(c''); \\ (e)+n::(g)+n;(g');(g'')+n:(h)+n, 1+\delta+2n;(h');(h''); \end{matrix} x, y, z\right] \\
&= F^{(3)}\left[\begin{matrix} (a)::(b);(b');(b''):(c), 1+\delta-\alpha-\beta;(c');(c''); \\ (e)::(g);(g');(g''):(h), 1+\delta-\alpha, 1+\delta-\beta;(h');(h''); \end{matrix} x, y, z\right]
\end{aligned} \tag{8}$$

Deductions. We give below some interesting expansions from the above result on specializing the parameters suitably:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(a)_n(-x)^n}{(\delta+n)_n(1+\delta-\alpha)_n} \\
&\quad \times F_A(a+n, 1+\delta-\beta+n, b_2, b_3; 1+\delta+2n, c_2, c_3; x, y, z) \\
&= F_A(a, 1+\delta-\alpha-\beta, b_2, b_3; 1+\delta-\alpha, c_2, c_3; x, y, z)
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(a_1)_n(-x)^n}{(\delta+n)_n(1+\delta-\alpha)_n} \\
&\quad \times F_E(a_1+n, a_1+n, a_1+n, 1+\delta-\beta+n, b_2, b_2; 1+\delta+2n, c_2, c_3; x, y, z) \\
&= F_E(a_1, a_1, a_1, 1+\delta-\alpha-\beta, b_2, b_2; 1+\delta-\alpha, c_2, c_3; x, y, z)
\end{aligned} \tag{10}$$

Further, setting $z=0$ in (9) we get

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(a)_n(-x)^n}{(\delta+n)_n(1+\delta-\alpha)_n}$$
$$\times F_2(a+n, 1+\delta-\beta+n, b_2; 1+\delta+2n, c_2; x, y)$$
$$=F_2(a, 1+\delta-\beta, b_2; 1+\delta-\alpha, c_2; x, y) \tag{12}$$

where F_2 is Appell's double hypergeometric function [11, p. 14].

3. Further generalizations of Theorem 1

In this section we give two generalizations of bilateral generating function (5) which are contained in the following theorems. Just as the generating function (5), Theorem 2 can also be deduced from the special case $r=3$ of Theorem 3 of [9].

Theorem 2. *Let $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$ be sequences of arbitrary complex numbers, then*

$$\sum_{n=0}^{\infty}\frac{(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}P_n^{(\alpha,\beta)}(w)\cdot\sum_{k,m,p=0}^{\infty}A_{n+k+m+p}B_{(m+p,m,p)}$$
$$\times C_{n+k+m}D_{n+k+p}E_{n+k}\frac{x^k y^m z^p}{(2+\alpha+\beta+2n)_k k!m!p!}$$
$$=\sum_{k,m,p=0}^{\infty}A_{k+m+p}B_{(n.+p,m,p)}C_{k+m}D_{k+p}E_k\frac{[\frac{1}{2}(1-w)x]^k y^m z^p}{(1+\alpha)_k k!m!p!} \tag{13}$$

provided each side has a meaning.

Theorem 3. *Let $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$ be sequences of arbitrary complex numbers, then*

$$\sum_{n=0}^{\infty}\frac{(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}P_n^{(\alpha,\beta)}(w)\sum_{k,m,p=0}^{\infty}A_{2n+2k+2m+p}$$
$$\times B_{2n+2k+m+p}C_{2n+2k+m}D_{(m+p,m,p)}E_{n+k}\frac{x^k y^m z^p}{(2+\alpha+\beta+2n)_k k!m!p!}$$
$$=\sum_{k,m,p=0}^{\infty}A_{2k+2m+p}B_{2k+m+p}C_{2k+m}D_{(m+p,m,p)}E_k\frac{[\frac{1}{2}(1-w)x]^k y^m z^p}{(1+\alpha)_k k!m!p!} \tag{14}$$

provided each side has a meaning.

Proof of Theorem 2. Following Srivastava and Pathan ([9], [10]) rather closely, we

$$\times A_{k+m+p}B_{(m+p,m,p)}C_{k+m}D_{k+p}E_k\frac{x^{k-n}y^mz^p}{(2+\alpha+\beta+2n)_{k-n}(k-n)!m!p!}$$

$$=\sum_{k,m,p=0}^{\infty}A_{k+m+p}B_{(m+p,m,p)}C_{k+m}D_{k+p}E_k\frac{x^ky^mz^p}{k!m!p!}$$

$$\times\Gamma(1+\alpha)\left[\sum_{n=0}^{\infty}\frac{(-k)_n(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(2+\alpha+\beta+n+k)}P_n^{(\alpha,\beta)}(w)\right]$$

which, in view of the expansion (4), gives

$$\Omega=\sum_{k,m,p=0}^{\infty}A_{k+m+p}B_{(m+p,m,p)}C_{k+m}D_{k+p}E_k\frac{[\frac{1}{2}(1-w)x]^ky^mz^p}{(1+\alpha)^kk!m!p!}$$

which is same as the right-hand side of (13). This evidently completes the proof of Theorem 2 under the assumption that various changes of the order of summations are permissible by the absolute convergence of the series involved. Thus in general, Theorem 2 holds for such values of the variables x, y, z and w for which each member of (13) exists. Theorem 2 may also be obtained directly from Theorem 3 of [9] by taking $r=3$.

Proof of Theorem 3 is similar to that of Theorem 2 and we, therefore, omit the details involved.

Deductions. If we assign suitable values to arbitrary sequences $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$ suitably, Theorem 2 can obviously be reduced to the general result (5) and its various special cases.

Similarly, by assigning suitable special values to the arbitrary coefficients $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$, Theorem 3 can be applied to deduce a number of bilateral generating functions involving Exton's triple hypergeometric functions [4].

Thus if we set

$$A_{2k+2m+p}=(a_1)_{2k+2m+p},\quad B_{2k+m+p}=C_{2k+m}=E_k=1$$

and

$$D_{(m+p,m,p)}=[(a_2)_p/(c_1)_{m+p}]$$

we obtain from (14):

$$\sum_{n=0}^{\infty}\frac{(a_1)_{2n}(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}X_1[a_1+2n,a_2;c_1,2+\alpha+\beta+2n;x,y,z]P_n^{(\alpha,\beta)}(w)$$
$$=X_1[a_1,a_2;c_1,1+\alpha;\tfrac{1}{2}(1-w)x,y,z]. \tag{15}$$

Similarly, the following bilateral generating functions are easily obtainable from Theorem 3 on assigning suitable values to the arbitrary coefficients:

$$\sum_{n=0}^{\infty}\frac{(a_1)_{2n}(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n}X_2[a_1+2n,a_2;2+\alpha+\beta+2n,c_2,c_3;x,y,z]P_n^{(\alpha,\beta)}(w)$$

$$\sum_{n=0}^{\infty} \frac{(a_1)_{2n}(-x)^n}{(1+\alpha)_n(1+\alpha+\beta+n)_n} X_7[a_1+2n, a_2, a_3; c_1, 2+\alpha+\beta+2n; x,$$

$$= X_7[a_1, a_2, a_3; c_1, 1+\alpha; \tfrac{1}{2}(1-w)x, y, z].$$

We conclude by observing that many other results may also be obt
Exton's triple hypergeometric functions X_8, X_{12}, X_{15}, X_{17} and X_{19}.

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Summability of Laguerre series at the point $x = 0$

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Abstract. In this paper, the authors prove a theorem on matrix summability of Laguerre series at the point $x=0$. Various results on Casàro, Nörlund and generalized Nörlund summability method have been deduced.

Keywords. Matrix summability; Laguerre series.

1. Introduction

The triangular matrix $[A]=\{\lambda_{n,k}\}$, where $n=0,1,2,3,\ldots$ and $k=0,1,2,3,\ldots$ and $\lambda_{n,k}=0$ for $k>n$, is regular (in the sense of defining a regular sequence to sequence transformation) if

$$\lim_{n\to\infty}\lambda_{n,k}=0,\quad \text{for every fixed } k \tag{1}$$

$$\sum_{k=0}^{n}|\lambda_{n,k}|\leqslant M,\quad \text{independent of } n \tag{2}$$

and

$$\lim_{n\to\infty}\sum_{k=0}^{n}\lambda_{n,k}=1. \tag{3}$$

A series $\Sigma_{n=0}^{\infty}u_n$ with the sequence of partial sums $\{S_n\}$ is said to be summable $[A]$ to the sum S, if

$$\sum_{k=0}^{n}\lambda_{n,k}S_k\to S,\quad \text{as } n\to\infty.$$

In the case

$$\lambda_{n,k}=\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}},\quad \beta\geqslant 0$$

where $A_n^{\beta-1}$ is determined by the identity

$$(1-x)^{-\beta}=\sum_{n=0}^{\infty}A_n^{\beta-1}x^n,\quad |x|<1,$$

the method $[A]$ reduces to the well-known Cesàro method (C,β). For

$$\lambda_{n,k}=\frac{p_{n-k}}{P_n},\quad P_n=p_0+p_1+\cdots+p_n(\leqslant 0),$$

the method $[A]$ reduces to the Nörlund method (N,p_n). Also for $\lambda_{n,k}=p_{n-k}q_k/R_n$, where $R_n=q_0p_n+q_1p_{n-1}+q_2p_{n-2}+\cdots+q_np_0(\neq 0)$, the method $[A]$ reduces to the well known generalized Nörlund method (N,p,q).

The Fourier Laguerre expansion of a function $f(x)\in L[0,\infty]$ is given by

$$f(x)\sim\sum_{n=0}^{\infty}a_n L_n^{(\alpha)}(x) \tag{4}$$

where

$$a_n=\left\{\Gamma(\alpha+1)\binom{n+\alpha}{\alpha}\right\}^{-1}\int_0^{\infty}e^{-y}y^{\alpha}f(y)\,L_n^{(\alpha)}(y)\,\mathrm{d}y \tag{5}$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of other $\alpha>-1$, defined by the generating function

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)w^n=(1-w)^{-\alpha-1}\exp\left(-\frac{xw}{1-w}\right)$$

and existence of the integral (5) is presumed in the sense of Lebesgue. Let us write

$$\phi(y)=\{\Gamma(\alpha+1)\}^{-1}e^{-y}y^{\alpha}\{f(y)-f(0)\}$$

M denotes a constant, which may not be the same at each of its occurrence.

2. Main result

In this short note we establish the following theorem:

Theorem. *Let the non-negative real sequence $\{\lambda_{n,k}\}$, be non-decreasing with respect to k. If for some suitable constants c and w (c and w are defined in lemma 3),*

$$\int_{c/n}^{w}\frac{|\phi(y)|\,\mathrm{d}y}{y^{(2\alpha+3)/4}}=o(n^{-(2\alpha+1)/4}),\quad \text{as } n\to\infty \tag{6}$$

and

$$\int_1^{\infty}e^{y/2}y^{-1/4}|\phi(y)|\,\mathrm{d}y<\infty, \tag{7}$$

then for $-1<\alpha<-1/2$, the series (4) is summable $[A]$ $(=[\lambda_{n,k}])$ at $x=0$ to the sum $f(0)$ provided that

$$\lim\sum^{n}\lambda_{n,k}=1 \tag{8}$$

Note. Since $\{\lambda_{n,k}\}$ is non-negative and non-decreasing in k, we have

$$(n-k)\lambda_{n,k} \leqslant \sum_{m=k+1}^{n} \lambda_{n,m} \leqslant 1. \tag{9}$$

Thus, for each fixed k,

$$\operatorname*{Lim}_{n\to\infty} \lambda_{n,k} = 0. \tag{10}$$

Hence from (8) and (10) it follows that $[A]$ is a regular method.

Lemma. For the proof of the theorem we need the following lemmas.

Lemma 1. *If* $\mu > -1$, *then as* $n \to \infty$,

$$\sum_{k=1}^{n} \lambda_{n,k} k^{\mu} = 0(n^{\mu}).$$

Proof. If $\mu \geqslant 0$ proof is obvious, so let $0 > \mu > -1$. Also let $V = [n/2]$, where $[\phi]$ denotes the integral part of ϕ.

Thus,

$$\begin{aligned}\sum_{k=1}^{n} \lambda_{n.k} k^{\mu} &= \sum_{k=0}^{n-1} \lambda_{n,n-k}(n-k)^{\mu} \\ &= \left(\sum_{k=0}^{V} + \sum_{k=V+1}^{n-1}\right)\lambda_{n,n-k}(n-k)^{\mu} \\ &= \Sigma_1 + \Sigma_2, \quad \text{say.}\end{aligned}$$

Now

$$\begin{aligned}|\Sigma_1| &\leqslant (n-V)^{\mu} \sum_{k=0}^{V} \lambda_{n,n-k} \\ &= O(n^{\mu})\end{aligned}$$

and

$$\begin{aligned}|\Sigma_2| &\leqslant \lambda_{n,n-V} \sum_{k=V+1}^{n-1} (n-k)^{\mu} \\ &= O(\lambda_{n,n-V} n^{\mu+1}) \\ &= O(n^{\mu}), \quad \text{by (9).}\end{aligned}$$

which proves the lemma.

Lemma 2. *From* (6) *we have*

$$\int_0^t |\phi(y)|\,\mathrm{d}y = \mathrm{o}(t^{\alpha+1}), \quad \text{as } t \to 0.$$

Proof. Working on the lines of Khare and Tripathi [1] we can prove the lemma.

Lemma 3. ([2] p. 175). *If α is real, c and w are fixed positive constants, then as $n \to \infty$.*

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-1/4} O(n^{\alpha/2-1/4}), & \text{if } \dfrac{c}{n} \leqslant x \leqslant w \\ 0(n^{\alpha}), & \text{if } 0 \leqslant x \leqslant \dfrac{c}{n} \end{cases}$$

Lemma 4. ([2], p. 239). *If α and λ are real constants, $a > 0$, $0 < \eta < 4$, then for $n \to \infty$*

$$\max \exp(-x/2) x^{\lambda} |L_n^{(\alpha)}(x)| \sim n^{Q}$$

where

$$Q = \begin{cases} \max\left(\lambda - \dfrac{1}{2}, \dfrac{\alpha}{2} - \dfrac{1}{4}\right), & \text{if } a \leqslant x \leqslant (4-\eta)n \\ \max\left(\lambda - \dfrac{1}{3}, \dfrac{\alpha}{2} - \dfrac{1}{4}\right), & \text{if } x \geqslant a \end{cases}$$

the maxima being taken in the intervals pointed out in the right hand members.

Proof of the theorem. The nth partial sum of the series (4) at $x = 0$ is given by (see [2])

$$S_n(0) - f(0) = \int_0^{\infty} \phi(y)\, L_n^{(\alpha+1)}(y)\, dy.$$

Therefore

$$I = \sum_{k=0}^{n} \lambda_{n,k}(S_k(0) - f(0)) = \sum_{k=0}^{n} \lambda_{n.k} \int_0^{\infty} \phi(y)\, L_k^{(\alpha+1)}(y)\, dy.$$

In order to prove the theorem, we must show that

$$I = o(1), \quad \text{as } n \to \infty.$$

We choose a positive integer $N(> w^{-1})$. We may assume $\lambda_{n,k} = 0$ for $k = 0, 1, 2, 3, \ldots,$ $N-1$.

Now, for $n > N$, we have

$$I = \sum_{k=N}^{n} \lambda_{n.k} \int_0^{\infty} \phi(y)\, L_k^{(\alpha+1)}(y)\, dy.$$

Let us write

$$I = \int_0^{c/n} + \int_{c/n}^{w} + \int_{w}^{n} + \int_{n}^{\infty}, \quad \text{where } c \text{ and } w \text{ are defined in lemma 3.}$$
$$= I_1 + I_2 + I_3 + I_4, \quad \text{say.}$$

Using lemma 3, we have

$$|I_1| \leqslant M \sum^{n} |\lambda_{n,k}| k^{\alpha+1} \int_0^{c/n} |\phi(y)|\, dy$$

$$= o(1), \quad \text{as } n \to \infty.$$

Again using lemma 3, we have

$$|I_2| \leqslant M \sum_{k=N}^{n} |\lambda_{n,k}| k^{(2\alpha+1)/4} \int_{c/n}^{w} \frac{|\phi(y)|\,dy}{y^{(2\alpha+3)/4}}$$

$$\leqslant M n^{(2\alpha+1)/4} o(n^{-(2\alpha+1)/4}) \quad \text{using (6) and lemma 1}$$

$$= o(1), \quad \text{as } n \to \infty.$$

Now I_3 can be written as

$$|I_3| \leqslant \sum_{k=N}^{n} |\lambda_{n,k}| \int_{w}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| e^{-y/2} \cdot y^{(2\alpha+3)/4} |L_k^{(\alpha+1)}(y)|\,dy.$$

Hence, by Lemma 4,

$$|I_3| \leqslant M \sum_{k=N}^{n} |\lambda_{n,k}| k^{(2\alpha+1)/4} \int_{w}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)|\,dy$$

$$\leqslant M n^{(2\alpha+1)/4} \int_{w}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)|\,dy.$$

Since $(2\alpha+3)/4 > 1/4$, we have

$$|I_3| \leqslant M n^{(2\alpha+1)/4} \int_{w}^{n} e^{y/2} y^{-1/4} |\phi(y)|\,dy$$

$$= O(n^{(2\alpha+1)/4}), \quad \text{from (7)},$$

$$= o(1), \quad \text{as } n \to \infty.$$

Finally, consider I_4

$$|I_4| \leqslant \sum_{k=N}^{n} |\lambda_{n,k}| \int_{n}^{\infty} e^{(y/2)} y^{-(3\alpha+5)/6} |\phi(y)| e^{(-y/2)} y^{(3\alpha+5)/6} \times |L_k^{(\alpha+1)}(y)|\,dy.$$

Using lemma 4, we get

$$|I_4| \leqslant M \sum_{k=N}^{n} |\lambda_{n,k}| k^{(\alpha+1)/2} \int_{n}^{\infty} \frac{e^{(y/2)} y^{-1/4} |\phi(y)|\,dy}{y^{(6\alpha+7)/12}}$$

$$\leqslant M n^{(\alpha+1)/2} n^{-(6\alpha+7)/12} \int_{n}^{\infty} e^{(y/2)} y^{-1/4} |\phi(y)|\,dy$$

$$= O(n^{(1/2)-(7/12)}), \quad \text{using (7)}$$

$$= o(1), \quad \text{as } n \to \infty.$$

This completes proof of the theorem.

Corollaries

COROLLARY 1

Let $\beta > 0$. If (6) and (7) hold good, then for $-1 < \alpha < -1/2$, the series (4) is summ *(C, β) at $x = 0$, to the sum $f(0)$.*

COROLLARY 2

Let the regular Nörlund method (N, p_n) be defined by a non-negative, non-incre *sequence $\{p_n\}$. If (6) and (7) hold good, then for $-1 < \alpha < -1/2$, the series* *summable (N, p_n) at $x = 0$ to the sum $f(0)$.*

COROLLARY 3

Let the regular generalised Nörlund method (N, p, q) be defined by a non-neg *non-increasing sequence $\{p_n\}$, and a non-negative non-decreasing sequence $\{q_n\}$.*

$$G(t) \equiv \int_0^t |\phi(y)|\,dy = o(t^{\alpha+1}) \quad \text{as } t \to 0,$$

and (7) holds good, then for $-1 < \alpha < -1/2$ the series (4) is summable (N, p, q) at *to the sum $f(0)$.*

COROLLARY 4

Let the regular generalised Nörlund method (N, p, q) be defined as in Corollary 3. *and (7) hold good, then for $-1 < \alpha < -1/2$, the series (4) is summable (N, p, q) at x* *to the sum $f(0)$.*

Proof of corollaries. If we put $\lambda_{n,k} = p_{n-k} q_k / R_n$, in our theorem, we get coroll

Putting $q_n = 1$ for all n, in corollary 4, we get corollary 2. If we set $p_n = \binom{n+\beta}{\beta-}$

$\beta > 0$, in corollary 2, we get corollary 1. To prove corollary 3, it is sufficient to that condition (11) implies condition (6). From condition (11), given $\varepsilon > 0$, we choose δ such that

$$|G(t)| \leqslant \varepsilon t^{\alpha+1}, \quad 0 < t \leqslant \delta.$$

Now

$$J = \int_{c/n}^{\delta} \frac{|\phi(y)|\,dy}{y^{(2\alpha+3)/4}} = M\left[\frac{G(y)}{y^{(2\alpha+3)/4}}\right]_{c/n}^{\delta} + M\int_{c/n}^{\delta} \frac{G(y)\,dy}{y^{(2\alpha+7)/4}}$$

where M is a constant, may be different at each occurrence. Hence

$$J \leqslant M\varepsilon + M\varepsilon n^{-(2\alpha+1)/4} + M\varepsilon \int_{c/n}^{\delta} y^{(2\alpha-3)/4}\,dy$$

or

$$\int_{c/n}^{\delta} \frac{|\phi(y)|\,dy}{y^{(2\alpha+3)/4}} = o(n^{-(2\alpha+1)/4}), \quad \text{as } n \to \infty.$$

Also

$$\int_{\delta}^{w} \frac{|\phi(y)|\,dy}{y^{(2\alpha+3)/4}} \text{ is a constant.}$$

Hence

$$\int_{c/n}^{w} \frac{|\phi(y)|\,dy}{y^{(2\alpha+3)/4}} = o(n^{-(2\alpha+1)/4}), \quad \text{as } n \to \infty.$$

This completes the proof of corollary 3.

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On the convergence properties of measurable multifunctions with values in a separable Banach space

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Abstract. In this paper we prove some convergence theorems for Banach space valued multifunctions. First we consider the notion of weak convergence of sets and prove a weak completeness and a weak compactness result of the Dunford–Pettis type for weakly compact, convex valued integrable multifunctions. Then we consider set valued martingales and establish two convergence theorems. One using the Kuratowski–Mosco mode of convergence and for the other the Hausdorff mode.

Keywords. Weak convergence of sets; set valued martingale; set valued Radon–Nikodym derivative; Kuratowski–Mosco convergence; support function; Hausdorff metric.

1. Introduction

In this note we prove some convergence theorems for measurable multifunctions with values in a separable Banach space, extending some recent results obtained by this author [13].

We consider the notion of weak convergence of multifunctions and prove a weak sequential completeness result and a weak sequential compactness result of the Dunford–Pettis type. These two results extend earlier ones proved by Artstein [1] for $\mathbb{R}^n$-valued multifunctions and by the author [13] for Banach space valued multifunctions. It should be noted that it was Artstein [1] who first introduced and studied the notion of weak convergence of multifunctions. Results of this nature proved to be useful in the sensitivity analysis of differential inclusions and control systems. For details we refer to Stassinopoulos–Vinter [15] (finite dimensional systems) and Papageorgiou [14] (for infinite dimensional systems). We also prove two convergence theorems for set valued martingales, extending the result of Neveu [10], who considered multifunctions with values in a separable dual Banach space and the results of Daures [5] and Hiai [7], who considered set valued martingales in $\mathbb{R}^n$.

2. Preliminaries

and

$$P_{(w)k(c)}(X)=\{A\subseteq X\text{: nonempty, } (w-) \text{ compact, (convex)}\}.$$

A multifunction $F:\Omega\to P_f(X)$ is said to be measurable, if for all $z\in X$ $\omega\to d(z,F(\omega))=\inf\{\|z-x\|:x\in F(\omega)\}$ is measurable. Other equivalent definitions of measurability can be found in Wagner [16]. By S_F^1 we will denote the collection of all selectors of $F(\cdot)$, that belong in the Lebesgue–Bochner space $L^1(X)$ i.e. $S_F^1=\{f\in L^1(X): f(\omega)\in F(\omega)\,\mu\text{-a.e.}\}$. Using Aumann's selection theorem (see Wagner [16], theorem 5.10), it is easy to see that $S_F^1\neq\emptyset$ if and only if $F(\cdot)$ is measurable and $\omega\to\inf\{\|x\|:x\in F(\omega)\}\in L_+^1$. Using S_F^1 we can define a set valued integral for $F(\cdot)$ by setting $\int_\Omega F(\omega)\mathrm{d}\mu(\omega)=\{\int_\Omega f(\omega)\mathrm{d}\mu(\omega): f\in S_F^1\}$. For $A\in 2^X\backslash\{\emptyset\}$ we set $|A|=\sup\{\|x\|:x\in A\}$ (the "norm" of the set A) and $\sigma(x^*,A)=\sup\{(x^*,x):x\in A\}x^*\in X^*$ (the "support function" of the set A). A multifunction $M:\Sigma\to P_{wkc}(X)$ is said to be a multimeasure (set valued measure), if for every $x^*\in X^*, A\to\sigma(x^*,M(A))$ is a signed measure.

Let Σ_0 be a sub-σ-field of Σ and $F:\Omega\to P_{f(c)}(X)$ a Σ-measurable multifunction *s.t.* $S_F^1\neq\emptyset$. Its set valued conditional expectation with respect to the sub-σ-field Σ_0, is defined to be the Σ_0-measurable multifunction $E^{\Sigma_0}F:\Omega\to P_{f(c)}(X)$ *s.t.* $S^1_{E^{\Sigma_0}F}=cl\, E^{\Sigma_0}S_F^1$, the closure taken in $L^1(X,\Sigma_0)$. If $|F(\cdot)|\in L_+^1$, then so does $|E^{\Sigma_0}F(\cdot)|$. For further details we refer to Hiai [7]. Let $\{\Sigma_n\}_{n\geqslant 1}$ be an increasing sequence of sub-σ-fields of Σ *s.t.* $V_{n\geqslant 1}\Sigma_n=\Sigma$. Let $F_n:\Omega\to P_f(X)$ be Σ_n-measurable multifunctions with $S^1_{F_n}\neq\emptyset$ for all $n\geqslant 1$. We say that $\{F_n,\Sigma_n\}_{n\geqslant 1}$ is a "set valued martingale" if and only if $E^{\Sigma_n}F_{n+1}(\omega)=F_n(\omega)\,\mu$-a.e.

Let $\{A_n,A\}_{n\geqslant 1}\subseteq P_f(X)$. We will say that the A_n's "weakly converge" to A, denoted by $A_n\xrightarrow{w}A$, if for all $x^*\in X^*\,\sigma(x^*,A_n)\to\sigma(x^*,A)$ as $n\to\infty$. Clearly the weak limit set A is unique up to convex hull. By $\mathscr{L}^1_{wkc}(X)$ we will denote all $P_{wkc}(X)$-valued multifunctions $F(\cdot)$ *s.t.* $\omega\to|F(\omega)|\in L_+^1$. As usual we identify multifunctions that differ on a μ-null set. Let $\{F_n,F\}_{n\geqslant 1}\subseteq\mathscr{L}^1_{wkc}(X)$. We will say that the F_n's converge weakly to $F(\cdot)$ in $\mathscr{L}^1_{wkc}(X)$, denoted by $F_n\xrightarrow{w}F$ in $\mathscr{L}^1_{wkc}(X)$, if for all $x^*(\cdot)\in L^\infty(X^*_{w^*})=L^1(X)^*$ (Dinculeanu–Foias theorem), we have

$$\int_\Omega\sigma(x^*(\omega),F_n(\omega))\mathrm{d}\mu(\omega)\to\int_\Omega\sigma(x^*(\omega),F(\omega))\mathrm{d}\mu(\omega).$$

Again we see that the limit is unique up to convex hull.

Another mode of set convergence that we will use and which has important applications, is the "Kuratowski–Mosco" convergence of sets. Let $\{A_n,A\}_{n\geqslant 1}\subseteq P_f(X)$ and set $s\text{-}\underline{\lim}\,A_n=\{x\in X:x=s\text{-}\lim x_n, x_n\in A_n, n\geqslant 1\}$ (here s-denotes the strong topology on X) and $w\text{-}\overline{\lim}\,A_n=\{x\in X:x=w\text{-}\lim x_{n_k}, x_{n_k}\in A_{n_k}, n_1<n_2<\cdots n_k<\cdots\}$ (here w-denotes the weak topology on X). We say that the A_n's converge to A in the Kuratowski–Mosco sense, denoted by $A_n\xrightarrow{K-M}A$, if $s\text{-}\underline{\lim}\,A_n=A=w\text{-}\overline{\lim}\,A_n$. From theorem 4.6 of [11], we know that if X is reflexive and $\sup_{n\geqslant 1}|A_n|<\infty$, then $K-M$ convergence of the A_n's implies weak convergence, while the converse is not in general true (see the remark following theorem 4.6 of [11]). In nonreflexive infinite dimensional Banach spaces, the two notions are not comparable, since the $K-M$ convergence is related to the epigraphical convergence of the support functions, which in general is different from the classical pointwise convergence.

3. Weak convergence results

First we prove a weak sequential completeness result in $\mathscr{L}^1_{wkc}(X)$. Our theorem extends proposition 4.8 of Artstein [1] and theorem 3.1 of [13]. Note that in theorem 3.1 of [13], we assumed that $F_n(\omega) \subseteq W \in P_{wkc}(X)\,\mu$-*a.e.* (pointwise boundedness). Here we employ a more general boundedness condition. In the section (Ω, Σ, μ) is a complete finite measure space.

Theorem 3.1. *If X, X^* are both separable, X has the RNP and $F_n : \Omega \to P_{wkc}(X)$ are measurable multifunctions s.t.*

(1) *$\{|F_n|\}_{n\geq 1}$ is uniformly integrable,*
(2) *$\cup_{n\geq 1} \int_A F_n(\omega)\,\mathrm{d}\mu(\omega)$ is relatively w-compact for all $A \in \Sigma$*
(3) *for every $x^* \in X^*, \{\sigma(x^*, F_n(\cdot))\}_{n\geq 1}$ is weakly Cauchy in $L^1(\Omega)$, then there exists $F \in \mathscr{L}^1_{wkc}(X)$ s.t. $F_n \xrightarrow{w} F$ in $\mathscr{L}^1_{wkc}(X)$.*

Proof. Let $k_n(A, x^*) = \int_A \sigma(x^*, F_n(\omega))\,\mathrm{d}\mu(\omega)$, $n \geq 1$. Because of hypothesis (3), for all $(A, x^*) \in \Sigma \times X^*, \lim k_n(A, x^*) = k(A, x^*)$ exists. From Nikodym's theorem, we know that $k(\cdot, x^*)$ is a signed measure. Set $L(A) = \overline{\mathrm{conv}}\,(\cup_{n\geq 1} \int_A F_n)$. Because of hypothesis (2) and the Krein–Smulian theorem, $L(A) \in P_{wkc}(X)$ for all $A \in \Sigma$. Also note that $|k(A, x^*)| \leq \sigma(x^*, L(A)) v \sigma(-x^*, L(A)) = \psi_A(x^*)$ and $\psi_A(\cdot)$ is m-continuous since $L(A) \in P_{wkc}(X)$ (here m-denotes the Mackey topology on X^* induced by the dual pair (X^*, X)). Invoking Hörmander's theorem [8], we get $M(A) \in P_{wkc}(X)$ s.t. $k(A, x^*) = \sigma(x^*, M(A))$. Then $M(\cdot)$ is a multimeasure, $M \ll \mu$ and $|M|(\Omega) < \infty$ (here $|M|(\cdot)$ denotes the total variation of $M(\cdot)$; see Hiai [7]). Thus invoking theorem 3 of Costé [4] we get $F \in \mathscr{L}^1_{wkc}(X)$ s.t. $M(A) = \int_A F(\omega)\,\mathrm{d}\mu(\omega)$ for all $A \in \Sigma$.

Let $x^*(\cdot) \in L^\infty(X^*) = L^1(X)^*$ (see Diestel–Uhl [6], theorem 1, p. 98), be countably valued; i.e. $x^*(\omega) = \Sigma_{k\geq 1} z_k^* \chi_{A_k}(\omega)$ with $z_k^* \in X^*$, $\|z_k^*\| \leq M$ and $A_k \in \Sigma, k \geq 1$. Let

$$s_n = \sum_{k\geq 1} \int_{A_k} \sigma(z_k^*, F_n(\omega))\,\mathrm{d}\mu(\omega), \quad s_n^m = \sum_{k=1}^{m} \int_{A_k} \sigma(z_k^*, F_n(\omega))\,\mathrm{d}\mu(\omega),$$

and

$$s = \sum_{k\geq 1} \int_{A_k} \sigma(z_k^*, F(\omega))\,\mathrm{d}\mu(\omega), \quad s^m = \sum_{k=1}^{m} \int_{A_k} \sigma(z_k^*, F(\omega))\,\mathrm{d}\mu(\omega).$$

Then clearly $s_n^m \to s^m$ as $n \to \infty$. Also

$$|s^m - s| = \left| \sum_{k=m+1}^{\infty} \int_{A_k} \sigma(z_k^*, F(\omega))\,\mathrm{d}\mu(\omega) \right| \leq \sum_{k=m+1}^{\infty} M \int_{A_k} |F(\omega)|\,\mathrm{d}\mu(\omega) \to 0$$

as $m \to \infty$.

Observe that $|s - s_n^{m(n)}| \to 0$ as $n \to \infty$. Also

$$|s_n^{m(n)} - s_n| = \sum_{k=m(n)+1}^{\infty} \int_{A_k} \sigma(z_k^*, F_n(\omega))\, d\mu(\omega)|$$

$$\leqslant \sum_{k=m(n)+1}^{\infty} M \int_{A_k} |F_n(\omega)|\, d\mu(\omega)$$

$$= M \int_{\bigcup_{k=m(n)+1}^{\infty} A_k} |F_n(\omega)|\, d\mu(\omega) \to 0 \text{ as } n \to \infty$$

since by hypothesis (1) $\{|F_n|\}_{n\geqslant 1}$ is uniformly integrable.

So $|s - s_n| \to 0$ as $n \to \infty$. Now recall (see Diestel–Uhl [6], corollary 3, p. 42), that every $x^*(\cdot) \in L^\infty(X^*)$ can be approximated in the $L^\infty(X^*)$-norm by countably valued $L^\infty(X^*)$-functions. Hence since $\int_A \sigma(x^*, F_n(\omega))\, d\mu(\omega) \to \int_A \sigma(x^*, F(\omega))\, d\mu(\omega)$ for all $(A, x^*) \in \Sigma x X^*$, by a simple density argument, we get

$$\int_\Omega \sigma(x^*(\omega), F_n(\omega))\, d\mu(\omega) \to \int_\Omega \sigma(x^*(\omega), F(\omega))\, d\mu(\omega) \text{ as } n \to \infty, \quad \text{for every}$$

$$x^*(\cdot) \in L^\infty(X^*)$$

$$\Rightarrow F_n \xrightarrow{w} F \text{ in } \mathscr{L}^1_{wkc}(X).$$ **QED**

Next we have a weak sequential compactness result of the Dunford–Pettis type. Our theorem extends, proposition 4.9 of Artstein [1] and theorem 4.1 of [13], where a pointwise boundedness hypothesis was used.

Theorem 3.2. *If X, X^* are both separable, X has the RNP and $F_n: \Omega \to P_{wkc}(X)$ are measurable multifunctions s.t.*

(1) $\{|F_n|\}_{n\geqslant 1}$ *is uniformly integrable,*
(2) *for every* $A \in \Sigma$, $\overline{\bigcup_{n\geqslant 1} \int_A F_n}^{w} \in P_{wk}(X)$, *then there exists a subsequence* $\{F_{n_k}\}_{k\geqslant 1}$ of $\{F_n\}_{n\geqslant 1}$ *s.t.* $F_{n_k} \xrightarrow{w} F$ *in* $\mathscr{L}^1_{wkc}(X)$.

Proof. Let D_0^* be a countable, strongly dense subset of X^*. Let $D^* = \{\sum_{k=1}^n \lambda_k z_k^*, n \geqslant 1, \lambda_k \in \mathbb{Q}, z_k^* \in D_0^*\}$. This is countable and so we can enumerate its elements; i.e. $D^* = \{z_k^*\}_{k\geqslant 1}$. From the classical Dunford–Pettis compactness criterion, we know that $\{\sigma(z_1^*, F_n(\cdot))\}_{n\geqslant 1}$ is relative weakly sequentially compact in $L^1(\Omega)$. So we can find a subsequence (denoted for economy in the notation by the same index) *s.t.* for all $A \in \Sigma$, $\int_A \sigma(z_1^*, F_n(\omega))\, d\mu(\omega) = \sigma(z_1^*, \int_A F_n) \to \phi(A, z_1^*)$. A new application of the Dunford–Pettis theorem, produces a further subsequence (denoted always by the same index) *s.t.* for all $A \in \Sigma \int_A \sigma(z_2^*, F_n(\omega))\, d\mu(\omega) = \sigma(z_2^*, \int_A F_n) \to \phi(A, z_2^*)$. We continue this way deriving sub-sub-subsequences. By a standard diagonal process, we produce a final subsequence of the original sequence *s.t.* for all $z^* \in D^* \int_A \sigma(z^*, F_n(\omega))\, d\mu(\omega) = \sigma(z^*, \int_A F_n) \to \phi(A, z^*)$.

Recalling that the support function is sublinear, for any $z^*, z^{*\prime} \in D^*$ we have:

$$\left|\sigma\left(z^*, \int_A F_n\right) - \sigma\left(z^{*\prime}, \int_A F_n\right)\right| \leqslant \sigma(z^* - z^{*\prime}, I_n(A))$$

e $L(A)=\overline{\text{conv}}[(\cup_{n\geqslant 1}\int_A F_n)\cup(-\cup_{n\geqslant 1}\int_A F_n)]\in P_{wkc}(X)$ (hypothesis (2) and the 1–Smulian theorem). So

$$|\phi(A,z^*)-\phi(A,z^{*\prime})|\leqslant\sigma(z^*-z^{*\prime},L(A))\leqslant|L(A)|\cdot\|z^*-z^{*\prime}\|.$$

us $\phi(A,\cdot)$ is uniformly continuous on D^*, which is strongly dense in X^*. Thus exists a unique strongly continuous extension $\hat{\phi}(A,\cdot)$ of $\phi(A,\cdot)$ on X^*. It is clear or all $z^*,z^{*\prime}\in D^*$ and for $\lambda\in\mathbb{Q}$, we have

$$\hat{\phi}(A,z^*+z^{*\prime})\leqslant\hat{\phi}(A,z^*)+\phi(A,z^{*\prime})$$

$$\hat{\phi}(A,\lambda z^*)=\lambda\hat{\phi}(A,z^*),A\in\Sigma.$$

en by a density argument, we deduce that $\hat{\phi}(A,\cdot)$ is sublinear. Furthermore since $z^*)|\leqslant\sigma(z^*,L(A))$ and $L(A)\in P_{wkc}(X)$, $\sigma(\cdot,L(A))$ is m-continuous and so $\hat{\phi}(A,\cdot)$ is ntinuous too. Using Hörmander's theorem, we get $M(A)\in P_{wkc}(X)$ *s.t.* $\hat{\phi}(A,x^*)=$ $M(A)),x^*\in X^*$.

w let $z^*\in X^*$. We can find $\{z^*_m\}_{m\geqslant 1}\subseteq D^*$ *s.t.* $z^*_m\xrightarrow{s}z^*$ in X^*. We have

$$\sigma\left(z^*_m,\int_A F_n\right)\to\phi(A,z^*_m)=\sigma(z^*_m,M(A))\text{ as }n\to\infty.$$

$$\phi(A,z^*_m)\to\hat{\phi}(A,z^*)\text{ as }m\to\infty.$$

om lemma 1.6, p. 50 of Attouch [2], we get

$$\phi\left(z^*_{m(n)},\int_A F_n\right)\to\hat{\phi}(A,z^*)\text{ as }n\to\infty.$$

en for any pair $(A,z^*)\in\Sigma x X^*$, we have

$$\begin{aligned}\left|\sigma\left(z^*,\int_A F_n\right)-\sigma(z^*,M(A))\right|&\leqslant\left|\sigma\left(z^*,\int_A F_n\right)\right.\\&\quad\left.-\sigma\left(z^*_{m(n)},\int_A F_n\right)\right|+\left|\sigma\left(z^*_{m(n)},\int_A F_n\right)-\sigma(z^*,M(A))\right|\\&\leqslant\sigma(z^*-z^*_{m(n)},L(A))+\left|\sigma\left(z^*_{m(n)},\int_A F_n\right)-\sigma(z^*,M(A))\right|\to 0\quad\text{as }n\to\infty.\end{aligned}$$

en by Nikodym's theorem, we get that $\sigma(z^*,M(\cdot))$ is a signed measure for every *. So $M(\cdot)$ is a multimeasure. Applying theorem 3 of Costé [4], we get $^1_{wkc}(X)$ *s.t.* $M(A)=\int_A F(\omega)\,d\mu(\omega)$. Then for every $(A,z^*)\in\Sigma x X^*$ we have

$$\int_A\sigma(z^*,F_n(\omega))\,d\mu(\omega)\to\int_A\sigma(z^*,F(\omega))\,d\mu(\omega).$$

in the proof of theorem 3.1, exploiting the density of countably valued functions $^\infty(X^*)=L^1(X)^*$, we get $F_n\xrightarrow{w}F$ in $\mathscr{L}^1_{wkc}(X)$. **QED**

4. Set valued martingales

In this section we prove two convergence theorems for set valued martingales. The first extends theorem 2 of Daures [5] and theorem 6.3 of Hiai [7] which deal with set valued martingales in $\mathbb{R}^n$. The second theorem, extends theorem 3 of Neveu [9], where the set valued martingale, takes values in a separable, dual Banach space.

Let $\{F_n, \Sigma_n\}_{n\geqslant 1}$ be a set valued martingale. A martingale selection of the F_n's is a sequence $f_n:\Omega\to X$ s.t. $\{f_n, \Sigma_n\}_{n\geqslant 1}$ is an X-valued martingale. We then write $\langle f_n\rangle\in MS(F_n)$. In this section (Ω, Σ, μ) is a complete probability space and X a separable Banach space, with x^* separable too.

Theorem 4.1. *If $F_n:\Omega\to P_{wkc}(X)$ are measurable multifunctions s.t.*

(1) $\{F_n, \Sigma_n\}_{n\geqslant 1}$ *is a set valued martingale,*

(2) $\overline{\cup_{n\geqslant 1} F_n(\omega)}^w \in P_{wk}(X)\mu$*-a.e.,*

(3) $\sup_{n\geqslant 1}|F_n(\omega)|\leqslant \phi(\omega)\mu$*-a.e. with* $\phi(\cdot)\in L^1_+$,
then there exists $F\in\mathscr{L}^1_{wkc}(X)$ *s.t.* $F_n(\omega)\xrightarrow{K-M}F(\omega)\mu$*-a.e.*

Proof. From corollary 2.3 of Luu [9], we know that we can find a sequence $\{(f_n^k)\}_{k\geqslant 1}$ of martingale selections of $\{F_n\}_{n\geqslant 1}$ (i.e. $\langle f_n^k\rangle\in MS(F_n)$ for all $k\geqslant 1$) s.t. $F_n(\omega)=\overline{\{f_n^k(\omega)\}_{k\geqslant 1}}$ for every $n\geqslant 1$ and every $\omega\in\Omega$. Let $W(\omega)=\overline{\cup_{n\geqslant 1}F_n(\omega)}^w$. From hypothesis (2), we know that for μ-almost all $\omega\in\Omega$, we have $W(\omega)\in P_{wk}(X)$. So invoking proposition 4.4 of Chatterji [3], we know that there exist $f^k\in L^1(X) k\geqslant 1$ s.t. $f_n^k(\omega)\xrightarrow{s} f^k(\omega)\mu$-a.e.. Let $F(\omega)=\overline{\text{conv}}\{f^k(\omega)\}_{k\geqslant 1}$. Clearly $F(\cdot)\in L^1_{wkc}(X)$ and for every $k\geqslant 1$ we have

$$f^k(\omega)\in s\text{-}\underline{\lim}\, F_n(\omega)\mu\text{-a.e.}$$

But the latter set is closed and convex. So we get

$$\overline{\text{conv}}\{f^k(\omega)\}_{k\geqslant 1}\subseteq s\text{-}\underline{\lim}\, F_n(\omega)\mu\text{-a.e.}$$

$$\Rightarrow F(\omega)\subseteq s\text{-}\underline{\lim}\, F_n(\omega)\mu\text{-a.e.} \tag{1}$$

On the other hand let $x^*\in X^*$. Invoking lemma 4 of Neveu [10], we have (since x^* is separable)

$$\sup_{k\geqslant 1}(x^*, f_n^k(\omega))\to\sup_{k\geqslant 1}(x^*, f^k(\omega)) \text{ for all } \omega\in\Omega\backslash N, \mu(N)=0$$

$$\Rightarrow\sigma(x^*, F_n(\omega))\to\sigma(x^*, F(\omega)) \text{ for all } \omega\in\Omega\backslash N, \mu(N)=0.$$

Invoking proposition 4.1 of [11] we get

$$w\text{-}\overline{\lim}\, F_n(\omega)\subseteq F(\omega)\mu\text{-a.e.} \tag{2}$$

From (1) and (2) above, we conclude that $F_n(\omega)\xrightarrow{K-M}F(\omega)\mu$-a.e. **QED**

Remark. (1) If $\dim X<\infty$, then $F_n(\omega)\xrightarrow{h}F(\omega)\mu$-a.e., where $h(\cdot,\cdot)$ is the Hausdorff metric

(2) It is clear from the above proof that $F_n(\omega) \to F(\omega)\,\mu$-a.e. and $F_n \to F$ in $\mathscr{L}_{wkc}(X)$.

We can strengthen the conclusion of theorem 4.1, if we assume more about the limit multifunction $F(\cdot)$, which by theorem 4.1 we know it exists in $\mathscr{L}^1_{wkc}(X)$.

Theorem 4.2. *If the hypotheses of theorem* 4.1 *hold, and* $F(\cdot)$ *takes values in a separable subset of* $(P_{wkc}(X), h)$, *then*

$$F_n(\omega) \xrightarrow{h} F(\omega), \text{ where } h(\cdot,\cdot) \text{ is the Hausdorff metric on } P_{fc}(X).$$

Proof. Let $K \in P_{wkc}(X)$ and let $\{x_m^*\}_{m \geqslant 1}$ be strongly dense in the closed unit ball B_1^* of X^*. Recalling that $E^{\Sigma_0}\sigma(x_m^*, F_n(\omega)) = \sigma(x_m^*, E^{\Sigma_0}F_n(\omega))\omega \in \Omega \backslash N, \mu(N) = 0, m \geqslant 1$ (see the Lemma in [12]), we see that $\{|\sigma(x_m^*, F_n(\omega)) - \sigma(x_m^*, K)|\}_{n \geqslant 1}$ is a positive submartingale for every $m \geqslant 1$. Invoking lemma 4 of Neveu [10], we get

$$\begin{aligned} h(F_n(\omega), K) &= \sup_{m \geqslant 1} |\sigma(x_m^*, F_n(\omega)) - \sigma(x_m^*, K)| \to \sup_{m \geqslant 1} |\sigma(x_m^*, F(\omega)) - \sigma(x_m^*, K)| \\ &= h(F(\omega), K)\,\mu\text{-a.e.} \end{aligned}$$

Let V be the separable subspace of $P_{wkc}(X)$ in which $F(\cdot)$ takes its values. Then by a simple density argument we get $h(F_n(\omega), K') \to h(F(\omega), K')$ for all $\omega \in \Omega \backslash N'$, $\mu(N') = 0$ and all $K' \in V$. Let $K' = F(\omega)$. We finally have

$$h(F_n(\omega), F(\omega)) \to 0\,\mu\text{-a.e.} \qquad \textbf{QED}$$

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Uniformly non-$l_n^{(1)}$ Musielak–Orlicz sequence spaces*

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Abstract. We give a necessary and sufficient condition for the uniformly non-$l_n^{(1)}$ property of Musielak–Orlicz sequence spaces l^Φ generated by a sequence $\Phi = (\phi_n : n \geqslant 1)$ of finite Orlicz functions such that $\lim_{u\to 0} u^{-1}\phi_n(u) = 0$ for each $n \in \mathbb{N}$. As a result, for $n_0 \geqslant 2$, there exist spaces l^Φ which are only uniformly non-$l_n^{(1)}$ for $n \geqslant n_0$. Moreover we obtain a characterization of uniformly non-$l_n^{(1)}$ and reflexive Orlicz sequence spaces over a wide class of purely atomic measures and of uniformly non-$l_n^{(1)}$ Nakano sequence spaces. This extends a result of Luxemburg in [19].

Keywords. Musielak-Orlicz sequence space; uniformly non-$l_n^{(1)}$; reflexivity; convexity.

1. Introduction

Convexity properties in Banach spaces and in particular, in Orlicz- and Musielak-Orlicz spaces have been studied by various authors. Akimovich [1] characterized superreflexive Orlicz spaces (see [4], [13] and [18] for definitions). B-convex Orlicz spaces (see [2], [7], [8] for the general notion of B-convexity) are described by Denker and Kombrink ([3]). Both results are generalized by Hudzik and Kamińska ([11]) to Musielak–Orlicz spaces.

The uniformly non-$l_n^{(1)}$ property (see [6]–[8]) for Orlicz spaces was studied in several papers ([9], [10], [23], [25]). We note that in this concrete case of a Banach function space it is possible to obtain more precise results than in general. In [10] it is shown that in the case of an infinite non-atomic measure, and in the case of a measure being a union of a countable number of atoms of unit mass, uniform non-squareness coincides with reflexivity and also with B-convexity. The same result is obtained in [25] for Orlicz spaces equipped with the Orlicz norm. The results in [10] were extended in [12] to Musielak–Orlicz spaces over a space of a non-atomic measure.

In this paper we investigate the uniformly non-$l_n^{(1)}$ property of Musielak–Orlicz sequence spaces. The situation will be different in general.

A function ϕ is called an Orlicz function, if $\phi: \mathbb{R} \to \mathbb{R}_+$ is convex, even and continuous

corresponding Musielak–Orlicz sequence space l^Φ is defined to be the set of all real sequences $x=(x_n)_{n\geqslant 1}$ such that for some $\lambda>0$ (depending on x)

$$I_\Phi(\lambda x)=\sum_{n=1}^{\infty}\phi_n(\lambda x_n)<\infty. \tag{1}$$

The norm in l^Φ is given by the Luxemburg norm:

$$\|x\|_\Phi=\inf\{\varepsilon>0: I_\Phi(\varepsilon^{-1}x)\leqslant 1\}. \tag{2}$$

A Musielak–Orlicz function $\Phi=(\phi_n)$ is said to satisfy the condition δ_2 if there are constants K, $a>0$ and a sequence $(c_n: n\geqslant 1)$ of positive numbers such that

$$\sum_{n=1}^{\infty} c_n<\infty, \tag{3}$$

and such that for every $n\geqslant 1$ and every $u\in\mathbb{R}$ satisfying $\phi_n(u)\leqslant a$

$$\phi_n(2u)\leqslant K\phi_n(u)+c_n. \tag{4}$$

Given a Musielak–Orlicz function $\Phi=(\phi_n)$ we shall denote by Φ^* the sequence (ϕ_n^*) of functions ϕ_n^* that are complementary to ϕ_n in the sense of Young, i.e.

$$\phi_n^*(v)=\sup\{u|v|-\phi_n(u): u>0\} \tag{5}$$

for $n=1,2,3,\ldots$ and $v\in\mathbb{R}$. It is well-known that $\Phi^{**}=\Phi$ (see [17]–[21]).

The functional $I_\Phi(x)=\sum_{n=1}^{\infty}\phi_n(x_n)$ is called the convex modular (see [20], [21], [22]), and it is called uniformly non-$l_n^{(1)}$ if there exists an $\varepsilon\in(0,1)$ such that for any $x^i\in l^\Phi$ with $I_\Phi(x^i)\leqslant 1\,(1\leqslant i\leqslant n)$

$$I_\Phi(n^{-1}(x^1\pm x^2\pm\cdots\pm x^n))\leqslant 1-\varepsilon \tag{6}$$

for some choice of signs. Uniformly non-$l_2^{(1)}$ is also called uniformly non-square.

Replacing I_Φ by the norm $\|\ \|_\Phi$ one obtains the properties uniformly non-$l_n^{(1)}$ and uniformly non-square for the Musielak–Orlicz space l^Φ.

Without loss of generality we may assume that $\phi_n(1)=1$ for all $n\in\mathbb{N}$. Otherwise define a new Musielak–Orlicz function $\psi=(\psi_n)_{n\geqslant 1}$ by $\psi_n(u)=\phi_n(a_n u)$ where $\phi_n(a_n)=1\,(n\geqslant 1)$. The operator $P: l^\Phi\to l^\psi$ defined by $(Px)_n=a_n^{-1}x_n\,(n=1,2,3,\ldots)$ is a modular isometry, i.e. $I_\psi(Px)=I_\Phi(x)$ ($x\in l^\Phi$), and an isometry between l^Φ and l^ψ. In §§2 and 3 we shall assume this property. In §2 we show that both functions Φ and Φ^* satisfy the δ_2-condition if and only if I_Φ and I_{Φ^*} are uniformly non-$l_m^{(1)}$ for some integer $m\geqslant 2$. In §3 we derive our main result: Φ and Φ^* satisfy the δ_2-condition if and only if l^Φ is uniformly non-$l_{m'}^{(1)}$ for any $m'\geqslant m$, where m is the smallest integer satisfying

$$\liminf_{n\to\infty}\phi_n(1-\varepsilon)>2^{-m+1} \tag{7}$$

for all sufficiently small $\varepsilon>0$.

If $\phi_n=b_n\cdot\rho$ where $b_n\geqslant 0$ and where ρ is a convex Orlicz function, we derive a characterization of reflexivity for the corresponding space l^Φ in §4. This extends

Finally we note that the δ_2-property for Φ and Φ^* is equivalent to reflexivity and B-convexity in all spaces considered in this note. (See the cited literature and the remark following Theorem 1.)

2. The uniformly non-$l_n^{(1)}$ property for the convex modular

We begin with some auxiliary lemmas. A Musielak–Orlicz function $\Phi = (\phi_n)$ is said to satisfy the δ_2^{ε}-condition where $\varepsilon > 0$, if Φ satisfies the δ_2-condition with respect to the constants K, $a > 0$ and $c_n \geqslant 0$ and if in addition $\sum_{n=1}^{\infty} c_n \leqslant \varepsilon$.

The proof of the first two lemmas is straightforward and omitted.

Lemma 1. *Let $\Phi = (\phi_n)$ be a Musielak–Orlicz function such that $\phi_n(u) > 0$ for all $u \neq 0$ and all $n \geqslant 1$. Then the following are equivalent:*

(1) Φ *satisfies the δ_2-condition.*

(2) Φ *satisfies the δ_2^{ε}-condition for every $\varepsilon > 0$.*

Moreover, the constant $a > 0$ can be chosen to be the same when passing from (1) *to* (2).

Let $\Phi = (\phi_n)$ be a Musielak–Orlicz function satisfying the condition δ_2 with constants $K, a > 0$ and (c_n). Choose $d_n > 0$ such that $\phi_n(d_n) = a (n \geqslant 1)$. Then there exists a sequence (b_n) of positive numbers such that $\sum_{n \geqslant 1} \phi_n(b_n) < \infty$ and for $u \in [b_n, d_n]$ we have $\phi_n(2u) \leqslant (K+1)\phi_n(u)$. (see [11]).

Lemma 2. *Using the above notation, for any $\eta > 1$ there exists a $\xi > 1$ such that for all $n \geqslant 1$ and all $u \in [b_n, d_n]$*

$$\phi_n(\xi u) \leqslant \eta \phi_n(u). \tag{8}$$

Lemma 3. *If a Musielak–Orlicz function Φ^*, complementary in the sense of Young to Φ, satisfies the condition δ_2 then for any $\varepsilon > 0$ there exists a $\delta \in (0, 1)$ and a sequence $\{h_n\}$ of positive numbers such that*

$$\sum_{n=1}^{\infty} \phi_n(h_n) \leqslant \varepsilon, \tag{9}$$

and

$$\phi_n(u/2) \leqslant (\delta/2)\phi_n(u), \tag{10}$$

for all $n \in \mathbb{N}$ and u satisfying $\phi_n(h_n) \leqslant \phi_n(u) \leqslant 1$.

Proof. Define

$$\psi_n(u) = \begin{cases} \phi_n(u) & \text{if} \quad |u| \leqslant 1 \\ u^2 & \text{if} \quad |u| > 1. \end{cases}$$

We have

$$\psi_n^*(u) = \sup_{v \geqslant 0} \{|u|v - \psi_n(v)\} = \sup \{|u|v - \phi_n(v)\} \tag{11}$$

and

$$\begin{aligned}\psi_n^*(u) &= \sup_{0 \leqslant v \leqslant |u|} \{|u|v - \psi_n(v)\} \\ &= \max\left\{ \sup_{0 \leqslant v \leqslant 2} \{|u|v - \psi_n(v)\},\ \sup_{2 \leqslant v \leqslant |u|} \{|u|v - v^2\} \right\} \quad (|u| \geqslant 2).\end{aligned}$$

Define $f_u(v) = |u|v - v^2$. f_u has a local maximum at $v = \frac{1}{2}|u|$ with $f_u(\frac{1}{2}|u|) = \frac{1}{4}u^2$. Hence for all $n \geqslant 1$ and $u \geqslant 2$

$$\psi_n^*(u) \geqslant \tfrac{1}{4}u^2. \tag{13}$$

Combining (12) and (13) we obtain

$$\psi_n^*(2u) \leqslant 4u^2 \leqslant 16\psi_n^*(u) \tag{14}$$

for all $n \geqslant 1$ and $|u| \geqslant 2$.

By assumption $u^{-1}\phi_n(u) \to 0$ as $u \to 0$ for every $n \geqslant 1$. Hence it follows that $\phi_n^*(u) = 0$ if and only if $u = 0$ for every $n \geqslant 1$. Applying Lemma 1, it follows that Φ^* satisfies the conditon $\dot{\delta}_2^\varepsilon$ for any $\varepsilon > 0$. Thus, there are $K, a > 0$ and sequences (b_n), (d_n) of positive numbers such that $b_n < d_n \leqslant 1$ $(n \geqslant 1)$ and such that $\sum_{n \geqslant 1} \psi_n^*(b_n) \leqslant \varepsilon/2$, $\psi_n^*(d_n) = a (n \geqslant 1)$ and

$$\psi_n^*(2u) \leqslant K\psi_n^*(u) \quad (n \geqslant 1, b_n \leqslant u \leqslant d_n). \tag{15}$$

Assume now that $u \in [d_n, 2]$. We have

$$\psi_n^*(2u) \leqslant \psi_n^*(4) \leqslant 16 = \frac{16}{a}\psi_n^*(d_n) \leqslant \frac{16}{a}\psi_n^*(u). \tag{16}$$

Combining (14), (15) and (16) we arrive at

$$\psi_n^*(2u) \leqslant \left(K \vee \frac{16}{a}\right)\psi_n^*(u) \quad (n \geqslant 1, |u| \geqslant b_n). \tag{17}$$

Applying Lemma 2 we conclude that there exists a $\xi > 1$ such that

$$\psi_n^*(\xi u) \leqslant 2\xi\psi_n^*(u) + 2\xi\psi_n^*(b_n) \quad (n \geqslant 1, u \in \mathbb{R}),$$

yielding $\psi_n^*(u) \geqslant (1/2\xi)\psi_n^*(\xi u) - \psi_n^*(b_n)$ and furthermore

$$\begin{aligned}\psi_n(u/2) &= \sup_{v \geqslant 0}\{(|u|/2)v - \psi_n^*(v)\} \leqslant \sup_{v \geqslant 0}\{(|u|/2\xi)\xi v - (1/2\xi)\psi_n^*(\xi v)\} + \psi_n^*(b_n) \\ &= (1/2\xi)\psi_n(u) + \psi_n^*(b_n).\end{aligned}$$

There exists a sequence $b_n' \geqslant 0$ such that $\psi_n^*(b_n) = \psi_n(b_n')$. For $n \geqslant 1$ and $u \geqslant (\sqrt{\xi} - 1)^{-1}$

Let $\tilde{b}_n = (\sqrt{\xi} - 1)^{-1} 2\xi b'_n$. By convexity of each ϕ_n, there is $\alpha \in (0,1)$ such tha $\sum_{n=1}^{\infty} \psi_n(\alpha \tilde{b}_n) \leqslant \varepsilon/4$. Define

$$g_n = \sup\left\{\frac{2\psi_n(u/2)}{\psi_n(u)} : \alpha\tilde{b}_n \leqslant u \leqslant b_n\right\}.$$

Note that $0 < g_n < 1$ for $n = 1, 2, \ldots$.

Let

$$A_k = \{n \in \mathbb{N} : g_n \leqslant 1 - k^{-1}\} \ (k = 1, 2, \ldots).$$

We have that $A_k \uparrow \mathbb{N}$ and applying the Beppo–Levi theorem we obtain tha $\sum_{n \in \mathbb{N}\setminus A_k} \psi_n(\tilde{b}_n) \to 0$ as $k \to \infty$.

Define $h_n = \alpha\tilde{b}_n \chi_{A_K} + \tilde{b}_n \chi_{\mathbb{N}\setminus A_K}$ where K is chosen in such a way that $\sum_{n\in\mathbb{N}\setminus A_K} \psi_n(\tilde{b}_n) \leqslant \varepsilon/$ Then $\sum_{n=1}^{\infty} \psi_n(h_n) \leqslant \varepsilon/2$. Moreover, for some $\delta \in (0,1)$ (namely $\delta = \max(\xi^{-1/2}, 1 - K^{-1}$ we have

$$\psi_n(u/2) \leqslant (\delta/2)\psi_n(u)$$

for all $n \in \mathbb{N}$ and $u \geqslant h_n$. Since $\psi_n(u) = \phi_n(u)$ for $|u| \leqslant 1$ this completes the proof. ∎

PROPOSITION 1

If Φ is a Musielak–Orlicz function such that $\Phi^* = (\phi_n^*)$ satisfies the δ_2-condition, t convex modular I_Φ is uniformly non-square in l^Φ.

Proof. By our assumptions and by the previous lemma there is a constant $\sigma \in (0,$ and a sequence (c_n) of positive numbers such that $\sum_{n=1}^{\infty} \phi_n(c_n) \leqslant 1/4$ and $\phi_n(u/2)$ $\{\sigma/2\}\phi_n(u)$ for all $n \in \mathbb{N}$ and $u \in [c_n, 1]$. Let $x, y \in l^\Phi$, $I_\Phi(x) \leqslant 1$, $I_\Phi(y) \leqslant 1$. If $I_\Phi(x) + I_\Phi(y)$ $7/4$, then $I_\Phi((x+y)/2) \leqslant 7/8 = 1 - (1/8)$. Now, assume $I_\Phi(x) + I_\Phi(y) > 7/4$. Define subset A of $\mathbb{N}$ by

$$A = \{n \in \mathbb{N} : \phi_n(x_n) + \phi_n(y_n) \geqslant 2\phi_n(c_n)\}.$$

We have $\sum_{n\in\mathbb{N}\setminus A} [\phi_n(x_n) + \phi_n(y_n)] \leqslant 2\sum_{n\in\mathbb{N}\setminus A} \phi_n(c_n) \leqslant \frac{1}{2}$. Hence

$$\sum_{n\in A} [\phi_n(x_n) + \phi_n(y_n)] > \tfrac{7}{4} - \tfrac{1}{2} = \tfrac{5}{4}.$$

We shall prove that for any $n \in A$, we have

$$\phi_n\left(\frac{x_n + y_n}{2}\right) + \phi_n\left(\frac{x_n - y_n}{2}\right) \leqslant \eta[\phi_n(x_n) + \phi_n(y_n)], \qquad (1$$

where $\eta \in (0,2)$ is independent of x and y. We shall consider two cases.

1. $\max[\phi_n(x_n), \phi_n(y_n)] \geqslant \phi_n(c_n)$ and $n \in A$. Then

$$\min\left[\phi_n\left(\frac{x_n+y_n}{2}\right), \phi_n\left(\frac{x_n-y_n}{2}\right)\right] \leqslant \phi_n\left(\frac{\max(|x_n|, |y_n|)}{2}\right)$$

$$\leqslant \frac{\sigma}{2}\phi_n(\max(|x_n|, |y_n|)) \leqslant \frac{\sigma}{2}\{\phi_n(x_n) + \phi_n(y_n)\}$$

Hence

$$\phi_n\left(\frac{x_n+y_n}{2}\right)+\phi_n\left(\frac{x_n-y_n}{2}\right)\leqslant\frac{\sigma+1}{2}\{\phi_n(x_n)+\phi_n(y_n)\}.$$

2. $\max[\phi_n(x_n),\ \phi_n(y_n)]<\phi_n(c_n)$ and $n\in A$. Then

$$\frac{\phi_n((x_n+y_n)/2)+\phi_n((x_n-y_n)/2)}{\phi_n(x_n)+\phi_n(y_n)}\leqslant\frac{\frac{1}{2}[\phi_n(x_n)+\phi_n(y_n)]+\phi_n(\max(|x_n|,|y_n|)/2)}{\phi_n(x_n)+\phi_n(y_n)}\leqslant\frac{3}{4}.$$

Thus, for any $n\in A$, we have

$$\phi_n\left(\frac{x_n+y_n}{2}\right)+\phi_n\left(\frac{x_n-y_n}{2}\right)\leqslant\left(\frac{3}{4}\vee\frac{\sigma+1}{2}\right)\{\phi_n(x_n)+\phi_n(y_n)\}.$$

Further, denoting $\eta=3/4\vee((\sigma+1)/2)$, we have

$$\begin{aligned}2-\left\{I_\Phi\left(\frac{x+y}{2}\right)+I_\Phi\left(\frac{x-y}{2}\right)\right\}&\geqslant I_\Phi(x)+I_\Phi(y)-\left\{I_\Phi\left(\frac{x+y}{2}\right)+I_\Phi\left(\frac{x-y}{2}\right)\right\}\\&\geqslant\sum_{n\in A}\phi_n(x_n)+\sum_{n\in A}\phi_n(y_n)-\sum_{n\in A}\phi_n\left(\frac{x_n+y_n}{2}\right)-\sum_{n\in A}\phi_n\left(\frac{x_n-y_n}{2}\right)\\&\geqslant(1-\eta)\sum_{n\in A}[\phi_n(x_n)+\phi_n(y_n)]\geqslant(1-\eta)\frac{5}{4}=\theta.\end{aligned}$$

Hence $\min(I_\Phi((x+y)/2),\ I_\Phi((x-y)/2))\leqslant 1-\theta/2$. ∎

Theorem 1. *Let $\Phi=(\phi_n)$ be a Musielak–Orlicz function. Then the following properties are equivalent:*

(*a*) *Both functions Φ and Φ^* satisfy the δ_2-condition*
(*b*) *I_Φ is uniformly non-square and I_{Φ^*} is uniformly non-$l_m^{(1)}$ for some integer $m\geqslant 2$*
(*c*) *There exists an integer $m_0\geqslant 2$ such that I_Φ and I_{Φ^*} are uniformly non-$l_m^{(1)}$ for all $m\geqslant m_0$.*

Remark. It is shown in [11] that reflexivity of l^Φ implies super-reflexivity, and B-convexity implies (*a*). Super-reflexivity also implies (*a*), since if Φ (or Φ^*) does not satisfy the δ_2-condition, then l^Φ (or l^{Φ^*}) contains an isometric copy of l^∞ (cf. [16]). The property (*a*), moreover, implies B-convexity of l^Φ and l^{Φ^*} (see [11]), and B-convexity implies reflexivity, since l^Φ is a Banach lattice (see [7]). Consequently one of the conditions (*a*), (*b*) or (*c*) is equivalent to reflexivity and B-convexity.

Proof. "(*a*)$\Rightarrow$(*b*)" By Proposition 1 I_Φ is uniformly non-square. By the above remark l^{Φ^*} is B-convex. Since $I_{\Phi^*}(x)\leqslant\|x\|_{\Phi^*}$ if $\|x\|_{\Phi^*}\leqslant 1$, it follows immediately that I_{Φ^*} is uniformly non-$l_m^{(1)}$ for some integer $m\geqslant 2$.
"(*b*)$\Rightarrow$(*c*)" is obvious.

"$(c) \Rightarrow (a)$" We show that Φ^* satisfies the δ_2-condition whenever I_Φ is uniformly non-$l_m^{(1)}$ for some integer $m \geqslant 2$. By symmetry, (a) follows.

Define the sequence $h_n = \sup\{u \in [0, d_n]: \phi_n(u/m) > (\sigma/m)\phi_n(u)\}$ $(n \geqslant 1)$ where $\phi_n(d_n) = \delta (n \geqslant 1)$, where $0 < \delta$, $\sigma < 1$ are fixed real numbers and where $m \geqslant m_0$ is an integer. Note that we put in the definition of h_n, $\sup \varnothing = 0$.

We shall prove that the condition

$$\sum_{n=1}^{\infty} \phi_n(h_n) < \infty \text{ for some } 0 < \delta, \sigma < 1 \tag{19}$$

is necessary to hold for I_Φ being uniformly non-$l_m^{(1)}$. Let us assume, therefore, that (19) does not hold. Then for every δ, $\sigma \in (0,1)$ there are a sequence (n_l) of positive integers and a sequence (u_l), $0 \leqslant u_l \leqslant h_{n_l} \leqslant d_{n_l}$, such that

$$\phi_{n_l}(u_l/m) > (\sigma/m)\phi_{n_l}(u_l) \quad (l = 1, 2, \ldots)$$

and

$$\sum_{l=1}^{\infty} \phi_{n_l}(h_{n_l}) = \infty.$$

There exists a subsequence (p_k) of (n_l) such that

$$\sum_{n_l = p_k + 1}^{p_{k+1}} \phi_{n_l}(u_l) \leqslant 1 \text{ and } \sum_{n_l = p_k + 1}^{p_{k+1}+1} \phi_{n_l}(u_l) > 1.$$

Obviously,

$$\sum_{n_l = p_k + 1}^{p_{k+1}} \phi_{n_l}(u_l) > 1 - \delta.$$

Define

$$x^k = \sum_{n_l = p_k + 1}^{p_{k+1}} u_l e_{n_l} \quad (k = 1, 2, \ldots, m).$$

Then $I_\Phi(x^k) \leqslant 1$ $(1 \leqslant k \leqslant m)$ and for every choice of signs ± 1 we obtain

$$I_\Phi(m^{-1}(x^1 \pm x^2 \pm \cdots \pm x^m)) = \sum_{k=1}^{m} \sum_{n_l = p_k + 1}^{p_{k+1}} \phi_{n_l}(u_l/m)$$

$$> \frac{\sigma}{m} \sum_{k=1}^{m} \sum_{n_l = p_k + 1}^{p_{k+1}} \phi_{n_l}(u_l) > \frac{\sigma}{m} \sum_{k=1}^{m} (1 - \delta) = \sigma(1 - \delta).$$

Since $\sigma(1-\delta)$ can be chosen arbitrarily close to one, I_Φ cannot be uniformly non-$l_m^{(1)}$.

By an argument similar to that used in the proof of Lemma 3 or in the proof of Lemma 1.2.3 in [11] it can be shown easily that [19] implies the δ_2-condition for Φ^*. ■

3. The uniformly non-$l_n^{(1)}$ property for $\|\cdot\|_\phi$

In this section we extend Theorem 1 replacing the convex modular by $\|\cdot\|_\Phi$. We show that (a) and (c) are still equivalent and we shall find the minimal m_0 for which $\|\cdot\|_\Phi$ is uniformly non-$l_{m_0}^{(1)}$. This is done in the following two propositions.

PROPOSITION 2

Assume that Φ satisfies the following condition:

$$(*)\begin{cases}\text{There exists a subsequence } N\subset\mathbb{N} \text{ and an integer } m\geqslant 2\\ \text{such that for all } \varepsilon>0\\ \lim\limits_{\substack{n\to\infty\\ n\in N}}\phi_n(1-\varepsilon)\leqslant 2^{-m+1}.\end{cases}$$

Then l^{Φ} is not uniformly non-$l_m^{(1)}$.

Proof. By assumption there exists a sequence $n_k\in\mathbb{N}$ $(k\geqslant 1)$ of integers such that for any $\eta>0$

$$\lim_{k\to\infty}\phi_{n_k}(1+\eta)=\infty.$$

Denote by e^l the n_lth unit vector in l^{Φ}. Now fix $k\geqslant 1$. For $l=0,\dots,m-1$ set

$$\tau_i(l)=\begin{cases}1 & j2^l\leqslant i<(j+1)2^l, 0\leqslant j\leqslant 2^{m-l}-2, j \text{ even}\\ -1 & j2^l\leqslant i<(j+1)2^l, 1\leqslant j\leqslant 2^{m-l}-1, j \text{ odd}\end{cases}$$

and define for $\lambda>0$

$$x(l)=(1-\lambda)\sum_{i=0}^{2^{m-1}-1}\tau_i(l)e^{k+i}.$$

Let now $\varepsilon_l\in\{\pm 1\}$ $(0\leqslant l\leqslant m-1)$ be arbitrary,

$$x=\frac{1}{m}\sum_{l=0}^{m-1}\varepsilon_l x(l).$$

We have

$$I_{\Phi}(x(l))=\sum_{i=0}^{2^{m-1}-1}\phi_{n_{k+i}}(1-\lambda)\leqslant 1 \text{ for } k \text{ large}$$

so that $\|x(l)\|_{\Phi}\leqslant 1$ $(0\leqslant l\leqslant m-1)$.

There exists $i_0\in\{0,\dots,2^{m-1}-1\}$ such that

$$\left|\sum_{l=0}^{m-1}\varepsilon_l\tau_{i_0}(l)\right|=m$$

and

$$|x_{n_{k+i0}}|=1-\lambda.$$

To see this, use induction over $m\geqslant 2$: It is clear for $m=2$. Assume that our claim holds for $m-1$. Then there exist $0\leqslant i<2^{m-1}$, $2^{m-2}\leqslant j<2^{m-1}$ such that

$$\sum_{l=1}^{m-1}\varepsilon_l x_{n_{k+i}}(l)=-\sum_{l=1}^{m-1}\varepsilon_l x_{n_{k+j}}(l)$$

and their absolute value equals $(m-1)(1-\lambda)$. Since one of these two numbers is

there exists a $\nu > 0$ such that for some choice $\varepsilon_l \in \{\pm 1\}$, $\|x\|_\Phi \leqslant 1 - \nu$. We may choose $\lambda > 0$ so small that for some $\eta > 0$

$$\frac{1-\lambda}{1-\nu} \geqslant 1 + \eta.$$

By the above construction, if k is large enough, we obtain for any choice of $\varepsilon_l \in \{\pm 1\}$ and for

$$x = \frac{1}{m} \sum_{l=0}^{m-1} \varepsilon_l x(l)$$

$$I_\Phi\left(\frac{x}{1-\nu}\right) = \sum_{n \geqslant 1} \phi_n\left(\frac{x_n}{1-\nu}\right) \geqslant \sum_{i=0}^{2^{m-1}-1} \phi_{n_{k+i}}\left(\frac{x_{n_{k+i}}}{1-\nu}\right)$$
$$\geqslant \phi_{n_{k+i0}}\left(\frac{1-\lambda}{1-\nu}\right) \geqslant \phi_{n_{k+i0}}(1+\eta).$$

If $k \to \infty$ we obtain a contradiction. ■

PROPOSITION 3.

Let Φ be a Musielak–Orlicz function such that I_Φ is, but $\|\cdot\|_\Phi$ is not uniformly non-$l_m^{(1)}$ for some integer $m \geqslant 2$. Then condition $(*)$ in Proposition 2 is satisfied.

Proof. We have by assumption:

$$** \begin{cases} \text{There exist } x^i(l) \in l^\Phi, \|x^i(l)\|_\Phi \leqslant 1, 1 \leqslant i \leqslant m, I \geqslant 1 \text{ and } \varepsilon > 0 \text{ such that} \\ \text{for some choice of } \varepsilon^i(l) \in \{\pm 1\} \\ \qquad I_\Phi\left(m^{-1} \sum_{i=1}^m \varepsilon^i(l) x^i(l)\right) \leqslant 1 - \varepsilon \quad (l \geqslant 1) \\ \text{and} \\ \qquad \lim_{l \to \infty} \inf_{\tau^i(l) \in \{\pm 1\}} \left\| m^{-1} \sum_{i=1}^m \tau^i(l) x^i(l) \right\|_\Phi = 1. \end{cases}$$

Denote by $x_n^i(l)$ the coordinates of $x^i(l)$ $(l, n \geqslant 1,\ 1 \leqslant i \leqslant m)$ given by the assumption and let $Z_n(l) = \max_{1 \leqslant i \leqslant m} |x_n^i(l)|$. Let us assume that condition $(*)$ does not hold. We shall produce a sequence $\tau^i(l) \in \{\pm 1\}$ such that the

$$x^\tau(l) = \frac{1}{m} \sum_{i=1}^m \tau^i(l) x^i(l) (l \geqslant 1) \text{ satisfy } \|x^\tau(l)\|_\Phi \leqslant 1 - \lambda \text{ for some } \lambda > 0.$$

Let $M = \{(n, l) \in \mathbb{N}^2 \colon \text{for all } \tau^i(l) \in \{\pm 1\}\ |x_n^\tau(l)| \leqslant Z_n(l) - \eta/m\}$ where $\eta > 0$ denotes some fixed number and where $x^\tau(l)$ is defined above. Then for $(n, l) \in M$ and $\phi_n(x_n^\tau(l)) \leqslant a$ by the δ_2-property there are $t > 1$ and $K \in \mathbb{R}, c_n \geqslant 0$ such that $\phi_n(t x_n^\tau(l)) \leqslant K \phi_n(x_n^\tau(l)) + c_n$. If $\phi_n(x_n^\tau(l)) > a$ then for some $t > 1$,

$$\phi_n(t x_n^\tau(l)) \leqslant \phi_n(Z_n(l)) \leqslant 1 \leqslant (1/a) \phi_n(x_n^\tau(l)).$$

Now let $(n,l)\notin M$. Then there exists $\tau^i(l)\in\{\pm 1\}$ such that $|x_n^\tau(l)|>Z_n(l)-\eta/m$. This implies that $|x_n^j(l)|>Z_n(l)-\eta$ for all $1\leqslant j\leqslant m$. Denote by

$$M_0=\{(n,l)\in\mathbb{N}^2\backslash M:\phi_n(x_n^\tau(l))\leqslant 2^{-m+1}+\xi_l \text{ for all } \tau^i(l)\in\{\pm 1\}\}$$

for some fixed sequence $\xi_l>0$, $\lim_{l\to\infty}\xi_l=0$. Then there exist $t>1$ and $K\in\mathbb{R}$ such that

$$\phi_n(tx_n^\tau(l))\leqslant K\phi_n(x_n^\tau(l))\quad (\forall (n,l)\in M_0, \forall\tau)$$

since otherwise condition $(*)$ holds.

Finally, let $M_1=\mathbb{N}^2\backslash(M\cup M_0)$, i.e. $(n,l)\in M_1$ implies $\phi_n(x_n^\tau(l))>2^{-m+1}+\xi_l$ for some τ, $|x_n^j(l)|>Z_n(l)-\eta$ $(1\leqslant j\leqslant m)$. Since $\phi_n(Z_n(l))>2^{-m+1}+\xi_l$ for fixed l, and since $\|x^i(l)\|_\Phi\leqslant 1$ for $1\leqslant i\leqslant m$, there are only finitely ($\leqslant m(2^{m-1}-1)$) n's with $(n,l)\in M_1$. Denote these n's by $n_1(l)<n_2(l)<\cdots<n_r(l)$ $(l\geqslant 1)$. Passing to a subsequence we may assume that for each $s=1,\ldots,r$

$$\lim_l \phi_{n_s}(l)(Z_{n_s}(l))=q_s$$

exists. Let $r_0\leqslant r$ be chosen so that $q_s=2^{-m+1}$ for $s\leqslant r_0$. Choosing $\xi_l\geqslant\max_{s\leqslant r_0}\phi_{n_s}(Z_{n_s}(l))-2^{-m+1}$ we see that w.l.o.g. $r_0=0$. Therefore we may assume that there exists a $\xi>0$ such that $(n,l)\in M_1$ implies $\phi_n(Z_n(l))\geqslant 2^{-m+1}+\xi$ and $|x_n^j(l)|\geqslant Z_n(l)-\eta$ $(1\leqslant j\leqslant m)$. We claim next that for some $\eta>0$ (as above)

$$\min_{1\leqslant j\leqslant m}\phi_n(x_n^j(l))\geqslant 2^{-m+1}-\xi/m\quad ((n,l)\in M_1).$$

Otherwise let the minimum be attained by $y_n(l)$. Then there exists a subsequence of $y_n(l)$ and $\eta_l>0$ such that $y_n(l)\geqslant Z_n(l)-\eta_l$ and $\phi_n(y_n(l))<2^{-m+1}-\xi/m$. If P denotes the subsequence of these n, we must have $1\leqslant\liminf y_n(l)$ and $\lim_{n\in P}\sup\phi_n(y_n(l))\leqslant 2^{-m+1}$, a contradiction.

It follows now that for fixed l

$$1\geqslant\frac{1}{m}\sum_{i=1}^m I_\Phi(x^i(l))\geqslant\sum_{(n,l)\in M_1}\frac{1}{m}\sum_{i=1}^m\phi_n(x_n^i(l))$$

$$\geqslant\sum_{(n,l)\in M_1}2^{-m+1}+\xi/m.$$

Therefore card $\{n:(n,l)\in M_1\}\leqslant 2^{m-1}-1$ and we find $\tau^i(l)\in\{\pm 1\}$ such that for each n with $(n,l)\in M_1$

$$|x_n^\tau(l)|\leqslant\frac{m-1}{m}Z_n(l)-(Z_n(l)-\eta)/m=\frac{m-2}{m}Z_n(l)+\eta/m\leqslant t^{-1}Z_n(l)$$

for some $t>1$, independent of n and l.

Now let $\varepsilon^i(l)$ denote the choice of signs in $(**)$. If for a subsequence of l's and n's $|x_n^\varepsilon(l)|\leqslant Z_n(l)-\eta'$ (for some $\eta'>0$), then for all $K>1$ there exists a $t>1$ such that

Since by Lemma 1, Σc_n can be chosen arbitrarily small, we choose K so that $K(1-\varepsilon)+\Sigma c_n \leqslant 1$. Therefore $\|x^\varepsilon(l)\|_\Phi \leqslant 1/t < 1$ for all l belonging to the subsequence, a contradiction. Consequently, for all l sufficiently large there exists n_l such that $|x^\varepsilon_{n_l}(l)| > Z_{n_l}(l) - \eta'$. It is not hard to see that $(n_l, l) \in M_1$, for otherwise we obtain a contradiction to (**). It follows now that

$$t|x^\tau_{n_l}(l)| \leqslant Z_{n_l}(l) \leqslant |x^\varepsilon_{n_l}(l)| + \eta'$$

and if $\eta' > 0$ is small enough for some $1 < t' < t$

$$t'|x^\tau_{n_l}(l)| \leqslant |x^\varepsilon_{n_l}(l)|.$$

For $s<1$ and $u \in \mathbb{R}$ with $\phi_n(u) \leqslant 1$, $\phi_n(su) \leqslant s\phi_n(u)$. (This follows easily from convexity and from the fact that ϕ_n vanishes at 0.) Putting $s = t'^{-1} = 1-\varepsilon'$ we arrive at

$$\begin{aligned}
I_\Phi(x^\tau(l)) &\leqslant \sum_{\substack{n=1\\ n\neq n_l}}^{\infty} \phi_n(x^\tau_n(l)) + \phi_{n_l}(t'^{-1}x^\varepsilon_{n_l}(l))\\
&\leqslant \frac{1}{m}\sum_{i=1}^{m}\sum_{n\neq n_l} \phi_n(x^i_n(l)) + (1-\varepsilon')\frac{1}{m}\sum_{i=1}^{m}\phi_{n_l}(x^i_{n_l}(l))\\
&\leqslant 1-\varepsilon'\frac{1}{m}\sum_{i=1}^{m}\phi_{n_l}(x^i_{n_l}(l))\\
&\leqslant 1-\varepsilon' m^{-1}2^{-m+1}.
\end{aligned}$$

Combining everything, we have constructed a sequence of signs $\tau^i(l) \in \{\pm 1\}$ such that for some $\tilde{\varepsilon} > 0$, $t > 1$ and $K \in \mathbb{R}_+$,

$$I_\Phi(x^\tau(l)) \leqslant 1 - \tilde{\varepsilon} \quad (l \geqslant 1),$$

$$\phi_n(tx^\tau_n(l)) \leqslant K\phi_n(x^\tau_n(l)) + c_n. \quad (l, n \geqslant 1).$$

It follows by an easy argument and by Lemma 1 that there exist $1 < t' < t$ such that

$$\sum_n \phi_n(t'x^\tau_n(l)) \leqslant 1 \quad (l \geqslant 1)$$

so that $\|x^\tau(l)\|_\Phi \leqslant 1/t'$, a contradiction. ■

Theorem 2. *Let $\Phi = (\phi_n)$ be a Musielak–Orlicz function and let $m \geqslant 2$ be an integer. Then the following conditions are equivalent:*

(a) both functions Φ and Φ^* satisfy the δ_2-condition and

$$\liminf_{n\to\infty} \phi_n(1-\varepsilon) > 2^{-m+1} \tag{20}$$

for all sufficiently small $\varepsilon > 0$.
(b) l^Φ is uniformly non-$l^{(1)}_m$.

Proof. It suffices to show that if I_Φ is uniformly non-$l^{(1)}_m$, so is $\|\ \|_\Phi$. First of all note

Let *us* assume that $\| \|_\Phi$ is not uniformly non-$l_m^{(1)}$. Then there exist $x_i(l) \in B_{l^\Phi}(1)$ $(1 \leqslant i \leqslant m, l \geqslant 1)$ such that

$$\lim_{l \to \infty} \inf_{\tau^i \in \{\pm 1\}} \left\| m^{-1} \sum_{i=1}^{m} \tau^i x^i(l) \right\|_\Phi = 1.$$

But, since I_Φ is uniformly non-$l_m^{(1)}$ there exist signs $\varepsilon^i(l)$ with

$$I_\Phi\left(\frac{1}{m} \sum_{i=1}^{m} \varepsilon^i(l) x^i(l) \right) \leqslant 1 - \varepsilon$$

for some $\varepsilon > 0$. Thus condition (**) holds and Proposition 3 contradicts our assumption that (*) does not hold. ■

Remark. Proposition 2 shows that there are Musielak–Orlicz sequence spaces l^Φ which are uniformly non-$l_{m'}^1$ only for $m' \geqslant m$ where $m \geqslant 2$ is some given integer.

COROLLARY 1.

Let $\Phi = (\phi_n)$ be a Musielak–Orlicz function such that for some $u > 0$, $C > 1/2$ and all sufficiently large n

$$1/2 < \phi_n(u) \leqslant C. \tag{21}$$

Then l^Φ is reflexive if and only if l^Φ is uniformly non-square.

COROLLARY 2.

Let $\Phi = (\phi_n)$ be a Musielak–Orlicz function satisfying the following condition:

$$(***)\begin{cases} \text{For any } \varepsilon \in (0,1) \text{ there is a } \delta = \delta(\varepsilon) > 1 \text{ such that} \\ \text{for } n \geqslant 1 \text{ and } u \in \mathbb{R}\ \phi_n(u) \leqslant 1 - \varepsilon \text{ implies} \\ \phi_n(\delta u) \leqslant 1. \text{ (see [15].)} \end{cases}$$

Then l^Φ is reflexive if and only if l^Φ is uniformly non-square.

Proof. Note that (***) implies that (**) cannot hold for any $m \geqslant 2$. Consequently (*) does not hold for any $m \geqslant 2$. ■

4. Orlicz sequence spaces over purely atomic measures

In this section we shall give a complete criterion for the uniform non-$l_n^{(1)}$ property of Orlicz sequence spaces l^ρ for a wide class of purely atomic measures, i.e. for Musielak–Orlicz sequence spaces l^Φ over a measure being a union of a countable family of atoms of measure one, where $\phi_n(u) = \rho(u) b_n$, b_n is the measure of the nth atoms and where ρ is a convex finite Orlicz function satisfying the condition $\rho(u)/u \to 0$ as $u \to 0$.

First, we shall prove the following

Lemma 4. Let $\Phi = (\phi_n)$ be a Musielak–Orlicz function such that $\phi_n(u) = b_n \rho(a_n u)$, where ρ is a convex Orlicz function and (a_n), (b_n) are sequences of positive numbers. The function Φ satisfies the condition δ_2 if and only if:

(i) *ρ satisfies the condition Δ_2 at infinity if $\Sigma_{n=1}^{\infty} b_n < \infty$, $\lim_n \inf [(\Sigma_{k=n+1}^{\infty} b_k)/b_n] > 0$ and ρ has finite values,*
(ii) *ρ satisfies the condition Δ_2 for all $u \in \mathbb{R}$ if there exists a subsequence (b_{n_k}) of (b_n) such that $\lim_k b_{n_k} = 0$ and $\Sigma_{k=1}^{\infty} b_{n_k} = \infty$,*
(iii) *ρ satisfies the condition Δ_2 at 0 if $0 < \lim_n \inf b_n < \infty$.*

Proof. If Φ satisfies the condition δ_2, then there are constants k, $\delta > 0$ and sequences $(c_n), (d_n)$ of positive numbers such that $c_n < d_n$, $\phi_n(d_n) = \delta$ for each $n \in \mathbb{N}$, $\Sigma_{n=1}^{\infty} \phi_n(c_n) < \infty$ and $\phi_n(2u) \leqslant k\phi_n(u)$ for $u \in [c_n, d_n], n = 1, 2, \ldots$ (see [11]). The last condition is equivalent to the following one $\rho(2a_n u) \leqslant k\rho(a_n u)$ for $a_n u \in [a_n c_n, a_n d_n]$, $n = 1, 2, \ldots$, equivalently

$$\rho(2u) \leqslant k\rho(u) \text{ for } u \in [a_n c_n, a_n d_n], \quad n = 1, 2, \ldots. \tag{22}$$

Now we shall consider the above three cases separately.

(i) Assume that ρ satisfies the condition Δ_2 at infinity, i.e. there are constants k, $a > 0$ such that $\rho(a) > 0$ and $\rho(2u) \leqslant k\rho(u)$ for all $|u| \geqslant a$. Define $c_n = a_n^{-1} a$. Then

$$\sum_{n=1}^{\infty} \phi_n(c_n) = \sum_{n=1}^{\infty} \rho(a) b_n < \infty.$$

Moreover, if $|u| \in [c_n, \infty)$, then $a_n |u| \geqslant a$, and so $\phi_n(2u) = b_n \rho(a_n 2u) \leqslant k b_n \rho(a_n u) = k\phi_n(u)$, and Φ satisfies the condition δ_2. The fact that condition δ_2 for Φ implies condition Δ_2 at infinity for ρ follows from the results in [12] and by Proposition 6 in [5].

(ii) We have $\rho(a_{n_k} d_{n_k}) = \delta b_{n_k}^{-1}$. Hence it follows that $\lim_k \rho(a_{n_k} d_{n_k}) = \infty$, and by the finiteness of ρ, we get $\lim_k a_{n_k} d_{n_k} = \infty$. Since $\Sigma_{n=1}^{\infty} \rho(a_n c_n) b_n < \infty$ and $\Sigma_{k=1}^{\infty} b_{n_k} = \infty$, there exists a subsequence (n_l) of the sequence (n_k) such that $\lim_l \rho(a_{n_l} c_{n_l}) = 0$. Since ρ^{-1} is continuous at zero and vanishes only at zero, this yields $\lim_l a_{n_l} c_{n_l} = 0$. Therefore by (22), $\rho(2u) \leqslant k\rho(u)$ for all $u \in \mathbb{R}$.

It is obvious that $\Phi = (\phi_n)$ satisfies condition δ_2 whenever ρ satisfies condition Δ_2 for all $u \in \mathbb{R}$.

(iii) We have $\rho(a_n d_n) = \delta b_n^{-1}$. By the assumptions about (b_n), we get $\lim_n \sup \rho(a_n d_n) > 0$. Since ρ is continuous at zero and vanishes only at zero, it is equivalent to $\lim_n \sup a_n d_n > 0$. By the condition $\Sigma_{n=1}^{\infty} \rho(a_n c_n) b_n < \infty$ it follows that $\lim_n \rho(a_n c_n) = 0$. This yields $\lim_n a_n c_n = 0$. Therefore, by (22), ρ satisfies the condition Δ_2 at zero.

Conversely, assume ρ satisfies the condition Δ_2 at zero, i.e. there are constants $k, a > 0$ such that

$$\rho(2u) \leqslant k\rho(u) \text{ for } u \in [0, a]. \tag{23}$$

i.e. $\phi_n(2u) \leqslant k\phi_n(u)$ whenever $\phi_n(u) \leqslant \lambda\rho(a)$ for $n = 1, 2, \ldots$. This means that Φ satisf
the condition δ_2 with $c_n = 0$ for $n = 1, 2, \ldots$.

Lemma 5. *Assume a Musielak–Orlicz function* $\Phi = (\phi_n)$ *is of the form* $\phi_n(u) = \rho(u)$ *for all* $n \in \mathbb{N}$ *and* $u \in \mathbb{R}$, *where* ρ *is a convex Orlicz function and* (b_n) *is a sequence positive numbers as in Lemma* 4, *cases* (*i*), (*ii*) or (*iii*). *If* Φ *satisfies the condition* δ *then it satisfies the condition* (***).

Proof. Since the proofs of (ii) and (iii) are similar we show only (i). Denote $\lambda = \sup_n$ and let $a > 0$ be such that $\rho(2a) \leqslant \varepsilon/2\lambda$. If $|u| \leqslant a$, then $\phi_n(2u) = \rho(2u)b_n \leqslant (\varepsilon/2\lambda)\lambda = \varepsilon$ Applying Lemmas 2 and 4, we get $\rho(\xi u) \leqslant (2-\varepsilon)/(2-2\varepsilon)\rho(u)$ for a constant $\xi \in (1,$ and each $u \geqslant a$. Hence, $\phi_n(u) \leqslant 1-\varepsilon$ for all $n \in \mathbb{N}$ implies $\phi_n(\xi u) \leqslant (2-\varepsilon)/(2-2\varepsilon)\, \phi_n(u) \leqslant$ for $n = 1, 2, \ldots$.

Theorem 3. *Let* Φ *be a Musielak–Orlicz function as in Lemma* 5 *and with finite valu Then the following assertions are equivalent*:

(*a*) *both functions* Φ *and* Φ^* *satisfy the condition* δ_2.
(*b*) *both functions* ρ *and* ρ^* *satisfy the*

- *condition* Δ_2 *at infinity, if* $\Sigma_{n=1}^{\infty} b_n < \infty$ *and* $\lim_n \inf ((\Sigma_{k=n+1}^{\infty} b_k)/b_n) > 0$,
- *condition* Δ_2 *for all* $u \in \mathbb{R}$, *if there exists a sub-sequence* (b_{n_k}) *of* (b_n) *such that* $\lim_k b_{n_k} =$ *and* $\Sigma_{k=1}^{\infty} b_{n_k} = \infty$,
- *condition* Δ_2 *at zero, if* $0 < \lim_n \inf b_n < \infty$,

(*c*) l^{Φ} *is uniformly non-square.*

Proof. We have $\phi_n^*(u) = \rho^*(u/b_n)b_n$ for $n \in \mathbb{N}$ and $u \in \mathbb{R}$. Thus, by Lemma 4, conditio (*a*) and (*b*) are equivalent. Further, by Lemma 5, Theorem 2 and Corollary 2, conditio (*a*), (*b*) and (*c*) are equivalent.

Note. In the first and second case of measures in condition (*b*) the assumption th the ϕ_n have finite values for all $n \in \mathbb{N}$ may be omitted, because it follows then fron suitable condition Δ_2. The equivalence of conditions (*a*), (*b*) and (*c*) holds without t assumption of finiteness of ϕ_n for all $n \in \mathbb{N}$, in all cases of measures considered he

Also note that condition (20) with $m = 2$ is trivially satisfied after normalization ϕ_n, since ρ is a finite Orlicz function. Hence (*a*) and (*c*) of the previous theorem a always equivalent.

Remark. In view of the remark after Theorem 1 we have obtained a characterizati of reflexive Orlicz sequence spaces over a wider class of purely atomic measures th in Theorem 5, p. 60 of Luxemburg [19].

5. Nakano sequence spaces

Musielak–Orlicz sequence spaces generated by a Musielak–Orlicz function $\Phi = (\phi$ such that $\phi_n = 1/p_n\ |u|^{p_n}$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$, where $p_n > 1$ for $n = 1, 2, \ldots$ are call Nakano sequence spaces. These spaces were investigated by Nakano in [22]. criterion for uniform convexity of these spaces was given by Sundaresan in [24] a

As a consequence of Theorem 2, we shall give a criterion ak_1 that these spaces are uniformly non-$l_n^{(1)}$ ($n \geqslant 2$, n integer). Since $\phi_n(2u) = 2^{p_n}\phi_n(u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$, Φ satisfies condition δ_2 if and only if $\sup_n p_n < \infty$. The complementary function $\Phi^* = (\phi_n^*)$ is of the form $\phi_n^*(u) = 1/q_n\,|u|^{q_n}$, where $1/p_n + 1/q_n = 1$ for $n = 1, 2, \ldots$. It is evident that $\sup_n q_n < \infty$ iff $\inf_n p_n > 1$. Thus, Φ^* satisfies condition δ_2 if and only if $\inf_n p_n > 1$. Note that if Φ satisfies the δ_2-condition, then $\lambda = \sup_n p_n < \infty$ and

$$\phi_n(\xi u) \leqslant \xi^\lambda \phi_n(u)$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$, i.e. Φ satisfies condition (***). Thus, by Theorem 2 and Corollary 2 we have:

Theorem 4. *For a Nakano sequence space l^Φ the following conditions are equivalent:*

(*a*) *both functions Φ and Φ^* satisfy the condition δ_2.*
(*b*) $1 < \inf_n p_n \leqslant \sup_n p_n < \infty$.
(*c*) *l^Φ is uniformly non-square.*

Remark. A Nakano sequence space is isomorphically isometric to the Musielak–Orlicz sequence space l^ψ generated by the Musielak–Orlicz function $\psi = (\psi_n)$ with $\psi_n(u) = |u|^{p_n}$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$, where $p_n > 1$ for all $n \in \mathbb{N}$. In fact, if $\Phi = (\phi_n)$ is the Musielak–Orlicz function determining the Nakano space, then taking

$$a_n = (p_n)^{-1/p_n},$$

we have $\phi_n(a_n) = 1$ for $n = 1, 2, \ldots$. We have $\psi_n(u) = \phi_n(a_n u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$. The operator P defined by $P_x = (a_n x_n)$ is an isometry from l^Φ onto l^ψ. We also remark that the conditions in the theorem are equivalent to reflexivity and B-convexity, among others.

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Congruence subgroup problem for anisotropic groups over semilocal rings

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Abstract. In Chapter I, a theorem of Margulis which gives the structure of normal subgroups of $SL(1, D)$ for a quaternion division algebra D over a global field K of characteristic not 2, is generalized to semi-local rings R in K. Using this, we obtain in Chapter II, a description of normal subgroups of $G(R)$ for K-anisotropic algebraic groups G of types $A_3, B_n, C_n, {}^1D_n, {}^2D_n$ and some forms of ${}^2A_n$. As a Corollary, a proof of the Platonov–Margulis conjecture is obtained for the above groups.

Keywords. Anisotropic algebraic group; global field; normal subgroup structure.

1. Introduction

Let K be a *global* field i.e. an algebraic number field or a field of algebraic functions in one variable over a finite field. Let S be a finite set of non-Archimedean valuations of K and R the semi-local ring

$$\{x \in K : x \in O_v \forall v \in S\}$$

where O_v is the ring of integers in the completion K_v of K at v. Our aim in this paper is to study the structure of normal subgroups of $G(R) = G(K) \cap GL(n, R)$ where G is an absolutely almost simple, simply-connected K-anisotropic (linear) algebraic subgroup of $GL(n)$. We first recall what is known over the global field itself. Let G be a K-simple, simple-connected algebraic group over a *global field* K. If G is K-isotropic, the simplicity of the factor group of $G(K)$ by its centre as an abstract group is equivalent to the Kneser–Tits conjecture, by results of Tits [19], and is known to be true due to results of Wang [25], Platonov–Janćevskii [10], and Platonov [8]. In the case when G is K-anisotropic, a conjecture of Platonov [9] states that $G(K)/Z(G(K))$ is simple if, and only if, the local groups $G(K_v)/Z(G(K_v))$ are simple for all non-Archimedean places v of K. (Here $Z(G(K))$ is the centre of $G(K)$). Equivalently: If G is a K-simple, simply-connected K-group, then $G(K)/Z(G(K))$ is simple as an abstract group if, and only if, G is K_v-isotropic for all non-Archimedean places v of K. It should be noted here that by the results of Margulis [6], it follows that any non-central normal subgroup N of $G(K)$ is of finite index. For a long time, the conjecture was known to be true only for the spin groups of non-degenerate

over K_v for all non-Archimedean places v since the number of variables is $\geqslant 5$). Th
for three-dimensional spin groups, that is when the number of variables is 3,
validity of the conjecture was proved by Margulis [7]. Margulis showed that w
K has characteristic $\neq 2$, any non-central normal subgroup N of $G(K)$ is open i
in the topology on $G(K)$ induced by mapping it diagonally in $\Pi_\Lambda G(K_v)$, where
the finite set of non-Archimedean places at which G is anisotropic; note that
implies Platonov's conjecture for three-dimensional spin groups.

Our results generalize Margulis' theorem in two directions: on the one hand,
treat all groups of the following types $- B_n, C_n (n \geqslant 2)$, ${}^1D_n, {}^2D_n (n \geqslant 4)$, ${}^1A_3$, form
${}^2A_3$ isotropic over all finite completions and forms of ${}^2A_n (n \geqslant 3)$ which split o
quadratic extensions; on the other hand, we allow for replacement of the field K
a semilocal ring in it. Also, it can be seen from Tits's classification [20] that, exc
for G of type A_n, it is always true that $G(K_v)$ is simple for all non-Archimedean pla
v of K. Thus the above conjecture of Platonov reduces in these cases to proving t
$G(K)/Z(G(K))$ is simple.

When K is an algebraic number field, the projective simplicity of $G(K)$ has b
obtained in the cases of G of types C_n, F_4, G_2, and certain forms of type A_n by Boro
[1], in the cases of G of types $B_n, C_n, G_2, F_4, E_7, E_8$ by Chernousov [2].

Chernousov [3] also shows that if the conjecture holds for A_3, it holds for all gro
of type ${}^1D_n, {}^2D_n (n \geqslant 4)$. Finally, Tomanov [21], [22], [23] has proved the conject
for the types $C_n (n \geqslant 2), D_n (n \geqslant 4), F_4, G_2$, all groups of type A_3 and in [24] dedu
the result for groups of type ${}^2A_{kd-1} (k \geqslant 2)$ assuming it for ${}^2A_{2d-1}$.

The statement of our main theorem (Theorem 0.2 and Theorem 10.1) is as follo

*Let K be any global field of characteristic $\neq 2$. Let G be an absolutely alm
simple, simply-connected, K-anisotropic, K-algebraic group of type A_1, ${}^1A_3$, or a f
of ${}^2A_3$ which is isotropic over all finite completions. $K_v, B_n, C_n (n \geqslant 2), {}^1D_n, {}^2D_n (n$
or a form of ${}^2A_l (l \geqslant 3)$ which splits over a quadratic extension of K. Let S be any fi
set of non-Archimedean places of K containing all those places where G is anisotro
Fix open compact subgroups Γ_v of $G(K_v) \forall v \in S$. Denoting by $pr: G(K) \to \Pi_S G(K_v)$,
diagonal mapping, and writing $\Gamma = pr^{-1} (\Pi_S \Gamma_v)$, Γ acquires a topology which we
the S-adic topology on Γ. Then, any non-central normal subgroup N of Γ is ope
the S-adic topology.*

For groups of type ${}^2A_3$ which remain anisotropic over a finite completion o
we can prove only a weaker result (Theorem 10.3).

The paper is divided into two parts. The main result is first established in the c
of 3-dimensional groups in the first chapter (Theorem 2) by arguments closely follow
those of Margulis [7]. The general case is proved in chapter 2 by using the resul
the special case and various embeddings of 3-dimensional groups. This, in particu
implies the truth of the conjecture for these groups in the case of any global field
of characteristic $\neq 2$. Taking into account Raghunathan's result [14] which compl
Margulis's theorem to characteristic 2 also, it appears likely that the result here,
hold in characteristic 2 also.

Chapter I. Three-dimensional groups

Notation:

In this chapter, we shall be using the following notations:

K := Global field of characteristic $\neq 2$
V := Set of valuations of K
D := A quaternion division algebra over $K, D_v := D \otimes_K K_v$ for $v \in V$
T := The (finite) set of valuations in V at which D ramifies
Λ := The set of finite valuations in T
δ := Reduced norm map from D to K as well as its extension $D_v \to K_v$, $v \in V$
$D^1 := \operatorname{Ker} \delta, D_v^1 := \{x \in D_v / \delta(x) = 1\}$ for any v in V.

If $pr_\Lambda: D \to \Pi_\Lambda D_v$ denotes the diagonal embedding. Margulis proved in [7].

Theorem 1. *For any non-central normal subgroup N of D^1, there exists on open norm subgroup W of $\Pi_\Lambda D_v^1$ of finite index, such that $N = pr^{-1}(W) \cap D^1$.*

In this chapter, we obtain a generalization of this theorem.

Let $S \subseteq V \setminus T$ be a fixed finite set of non-Archimedean valuations. Then, for all in S, we can identify D_v with $M_2(K_v)$.

Fix such an identification for each $v \in S$ and consider the diagonal embeddi $pr: D \to \Pi_\Lambda D_v \times \Pi_S M_2(K_v)$. We study the ring

$$R = pr^{-1}(\Pi_\Lambda D_v \times \Pi_S M_2(\mathcal{O}_v)).$$

(In the above $\mathcal{O}_v$ denotes the local ring of integers of K_v).

If σ denotes the canonical involution of D, then, $\sigma(x) = \mathrm{Tr}_{\mathrm{red}},(x) - x$ so that σ leav $M_2(\mathcal{O}_v)$ fixed for all $v \in S$. Let R^* denote the multiplicative group of invertible elemen in R and $R^1 = \{x \in R^*: \delta(x) = 1\}$. The generalization of Margulis's theorem, referr to above, is the following.

Theorem 2. *For any non-central normal subgroup N of R^1, there exists on open norn subgroup W of finite index in $\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v)$ such that $N = pr^{-1}(W) \cap R^1$.*

According to Margulis [6], a non-central subgroup N in R^1 has finite index in Let r be the index. Then $x^r \in N$ for all $x \in R^1$. Clearly $N' :=$ Group generated $\{x^r / x \in R^1\}$, is a normal subgroup of R^* and $N' \subset N$; also N' is not central in Thus, there is no loss of generality in assuming in Theorem 2 above that N normalized by R^*. The proof closely follows that of Margulis.

1. Results on the reduced norm δ

Lemma 1.1. *For any finite set $A \subseteq V \setminus (T \cup S)$, under the diagonal embedding $pr_A: R^1$ $\Pi_A D_v^1$, the image of N is dense in $\Pi_A D_v^1$.*

Proof. Since N has finite index in R^1, and R^1 is dense in $\Pi_A D_v^1$, the result follo from the fact that for $v \in V \setminus (T \cup S)$, D_v^1 has no open subgroup of finite index.

Lemma 1.2. *For any v in V, and $g \in GL_2(K_v)$, the set $\{h \in SL_2(K_v)$: gh is diagonaliza over $K_v\}$ contains a non-empty open set of $SL_2(K_v)$.*

Remark 1.3. Since $R = pr^{-1}(\Pi_\Lambda D_v \times \Pi_S M_2(\mathcal{O}_v))$, it follows that $R^* = pr^{-1}(\Pi_\Lambda D_v^*$ $\Pi_S GL_2(\mathcal{O}_v))$, and the $R^1 = pr^{-1}(\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v))$. Define $K_0 = \{x \in \delta(R^*)/x \in$ $\forall v \in \Lambda \cup S\}$. For any x in D, we denote by $K(x)$ the smallest subfield of D containing

Lemma 1.4. *Let Ψ be a finite set of maximal subfields of D, and let $B \subseteq K_0$ a fir set. For any g in R^*, there exists $n \in N$ such that gn does not belong to K and such t $B \subseteq \delta(L^* \cap R^*) \cdot \delta(K(gn) \cap R^*) \forall\, L \in \Psi$.*

For the proof of this lemma, we use the following:

Lemma 1.5. *Suppose $q: K^r \to K$ is a non-degenerate quadratic form in r variables, wh is isotropic. Let $S \subset V$ be a finite set and suppose $w_v \in K_v^r (v \in S)$ are non-zero isotro vectors. Then there exists a sequence $\{x_n\} \in K^r$ of isotropic vectors such that $\{x_n\} \to$ in K_v^r for every $v \in S$.*

Proof. Firstly, we claim that there exists $x \in K^r$ such that $x \neq 0$, $q(x) = 0$ a $q(x + w_v) \neq 0 \forall v \in S$. For this, we write $K^r = H \perp H'$ where H is a hyperbolic pla $\langle x_1, x_2 \rangle$, that is, $q(x_1) = 0 = q(x_2)$ and $q(x_1, x_2) = 1$ (Here we write q also for bilinear form which is associated to q).

Let $\{y_1, y_2, \ldots, y_{r-2}\}$ be a basis of H' over K. Since H is universal, we can find for $1 \leqslant i \leqslant r-2$ and $j = 1, 2$ such that $-q(y_i) = q(\alpha_{i1} x_1 + \alpha_{i2} x_2)$, $1 \leqslant i \leqslant r-2$.

Then the vectors $\{x_1, x_2, y_i - \alpha_{i1} x_1 - \alpha_{i2} x_2\}$ are isotropic and form a basis of that is, we have shown that there exists a basis of K^r consisting of isotropic vect for q. Rename this basis $\{x_1, x_2, \ldots, x_r\}$. For each $v \in S$, at least one x_i satisf $q(x_i, w_v) \neq 0$ for, otherwise, $q(y, w_v) = 0$ for every $y \in K_v$ which contradicts the no degeneracy of q.

Let $\Omega_v = \{x \in K^r / q(x, w_v) \neq 0\}$. Then Ω_v is non-empty and Zariski open in K^r. Th $\cap_{v \in S} \Omega_v$ is Zariski open and non-empty. So, we have shown the truth of the first cla made that there exists $x \in K^r$ such that $q(x) = 0$ and $q(x, w_v) \neq 0$ for each v in S. Ta a sequence $\{y_n\} \in K^r$ converging to $w_v - x \in K_v^r$. We can assume that $q(y_n) \neq 0$ for al

Choose $t_n = -2.q(x, y_n) q(y_n)^{-1}$; then we have isotropic vectors $x_n = x + t_n y_n$ wh converge to w since $\{t_n\} \to 1$.

Proof of Lemma 1.4. Let $\Theta = \{v \in V / B \subseteq \delta(L_v) \forall\, L \in \Psi\}$. Now $\delta(R^*) \subseteq \{x \in K$ $x \in K_v^2 \forall v \in T \cap V_\infty\}$, where $V_\infty =$ Set of infinite valuations in V. Thus $K_0 = \{x \in \delta(R$ $x \in K_v^2$ for all $v \in \Lambda \cup S\} \subseteq \{x \in K^* / x \in K_v^2 = \delta(K_v) \forall v \in S \cup T\}$.

Therefore $S \cup T \subseteq \Theta$. On any $L \in \Psi, \delta/L$ is a quadratic form in two variables o K, and represents over K_v, for almost all v, all the elements of the finite set B. He $V \backslash \Theta$ is finite, and contained in $V \backslash (T \cup S)$. Moreover $B \subset \delta(L_v \cap R_v^*)$ for $v \in \Lambda \cup S$

By lemma 1.1 and 1.2, for the element $g \in R^*$, there exists $n \in N$ with gn not in and with gn diagonalizable over K_v for all $v \in V \backslash \Theta$.

So, we have $\delta(K(gn)_v) = K_v \forall v \in V \backslash \Theta$. Thus, $\forall b \in B$, $\forall L \in \Psi$, the equation $\delta(l) = b.\delta$ has non-zero solutions (l_v, x_v) in $L_v \times K(gn)_v$ for all v in V. By the Hasse–Minkow theorem, there exists a non-zero solution $(l, x) \in L \times K(gn)$. Also for $v \in S \cup \Lambda$ we c find $l_v \in K_v \cap R_v^*$ such that $\delta(l_v) = b$ so that $(l_v, 1)$ is a solution of the equation $\delta(x) -$

which belongs to each of the open neighbourhoods U_v of $(l_v, 1)$, where $U_v = (L_v \cap GL_2(\mathcal{O}_v)) \times (K(gn_v \cap GL_2(\mathcal{O}_v))$ for all $v \in S$. Thus, $(l, x) \in (L^* \cap R^*) \times (K(gn)^* \cap R^*)$ satisfies $\delta(l) = b.\delta(x)$.

COROLLARY 1.6

For any finite set $B \subseteq K_0$, any integer $t \geqslant 2$, and any $g \in R^*$, there exist $n_1, n_2, \ldots, n_t \in N$ such that gn_i is not in K and $B \subseteq (K(gn_i^* \cap R^*) \cdot (K(gn_j^* \cap R^*)$ for all $1 \leqslant i \leqslant j \leqslant t$.

Proof. Induction on t.

2. Some definitions and a lemma on σ

For any $v \in V \setminus V_\infty$, there exists an open subgroup Φ_v of finite index in K_v^* (respectively $\mathcal{O}_v^*$) for v not in S (respectively v in S), such that 1 does not belong to Φ_v.

Let $W =$ Closure of image $(N \to \Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v))$.

Write $\Phi = \Pi_\Lambda \cup_S \Phi_v$ and $W_0 = W \cdot \Phi$.

Let $F = pr^{-1}(W) \cap R^1$, and $F_0 = pr^{-1}(W_0) \cap R^*$.

Define τ: $D^* \to D^1$ by $\tau(x) = x \cdot \sigma(x)^{-1} = x^2 \cdot \delta(x)^{-1}$.

Note that $\tau(R^*) \subseteq R^1$ by Remark 1.3. We have $\tau(F_0) \subseteq F$ and $F_0 \cap R^1 = F$. Since R^1/N is finite, W is a closed normal subgroup of finite index in $\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v)$; hence it is open too. Therefore, W_0 is an open normal subgroup of finite index in $\Pi_\Lambda D_v^* \times \Pi_S GL_2(\mathcal{O}_v)$. For $g \in R^*$, set $C(g) = \{h \in R^* : [g, h] \in N\}$, the centraliser of g modulo N. Since N is normalized by R^*, $C(g)$ is a subgroup. With these notations, Theorem 2 amounts to showing $F = N$.

Lemma 2.1. *For any $x \in R$, there is a corresponding $y \in R^*$ such that $\sigma(x) = y.x.y^{-1}$.*

Proof. We wish to find $y \in R^*$ such that $\mathrm{Tr}(y) = 0$ and $\mathrm{Tr}(yx) = 0$, where $\mathrm{Tr}: D \to K$ denotes the reduced trace. For, then $y + \sigma(y) = 0 = yx + \sigma(yx)$, since $\sigma(d) = \mathrm{Tr}(d) - d \,\forall d \in D$ and so $yx + \sigma(x)\sigma(y) = 0$, that is, $yx = \sigma(x)y \Rightarrow \sigma(x) = yxy^{-1}$.

So, consider the two-dimensional space

$$\{y \in D : \mathrm{Tr}(y) = 0 \text{ and } \mathrm{Tr}(yx) = 0\}.$$

This is dense in

$$\prod_{1 \leqslant i \leqslant r} \{y_i \in D_{v_i} : \mathrm{Tr}(y_i) = 0, \mathrm{Tr}(y_i x) = 0\}$$

and hence, has to intersect the open set

$$\prod_{1 \leqslant i \leqslant r} \{y_i \in GL_2(\mathcal{O}_{v_i}) : \mathrm{Tr}(y_i) = 0, \mathrm{Tr}(y_i x) = 0\}$$

non-trivially, provided this last open set is non-empty. Let us show this non-emptiness $\forall v \in S$.

Let $v \in S$. Write $x = \begin{pmatrix} a & b \\ \end{pmatrix}$ as an element in $D_v = M_2(K_v)$

We want $y_v = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL_2(\mathcal{O}_v)$ such that $\mathrm{Tr}(y_v) = 0 = \mathrm{Tr}(y_v x)$. It is trivially chos
if x is a scalar matrix; so assume x to be non-scalar. Moreover, we can assume x a non-scalar matrix mod(Π_v), the maximal ideal of $\mathcal{O}_v$, for, we can write $x =$ scal matrix $+ x'\Pi_v^k$ in $M_2(\mathcal{O}_v)$ with $k \geqslant 0$ and x' a non-scalar matrix mod(Π_v) and, t choice of y_v which works for x' also does for x.

The conditions $\mathrm{Tr}(y_v) = 0 = \mathrm{Tr}(y_v x)$ are equivalent to

$$r, s, t, u \in \mathcal{O}_v, u = -r,$$

$$(a-d)r + cs + bt = 0,$$

$$r^2 + st \in \mathcal{O}_v^*.$$

Case 1: If b (respectively c) is in $\mathcal{O}_v^*$, then one can choose

$$y_v = \begin{pmatrix} 1 & 0 \\ b^{-1}(d-a) & -1 \end{pmatrix}$$

$$\left[\text{respectively} \quad y_v = \begin{pmatrix} 1 & c^{-1}(d-a) \\ 0 & -1 \end{pmatrix}\right] \in GL_2(\mathcal{O}_v).$$

Case 2: If neither b nor c is a unit, then $a - d \in \mathcal{O}_v^*$, since x mod(Π_v) is a non-scala and one can choose

$$y_v = \begin{pmatrix} -(b+c)(a-d)^{-1} & 1 \\ 1 & (b+c)(a-d)^{-1} \end{pmatrix} \in GL_2(\mathcal{O}_v).$$

Remark 2.2. If $x \in R^1$ and $(1+x)/2 \in R^*$, then $\tau((1+x)/2) = x$. Thus, since W_0 is op in $\Pi_\Lambda D_v^* \times \Pi_S GL_2(\mathcal{O}_v), \exists$ a neighbourhood $U \subseteq W$ of the identity in $\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}$ such that $\tau(F_0) \supseteq pr^{-1}(U) \cap R^1$. Call $F_U = pr^{-1}(U) \cap R^1$. Note that $F = F_U \cdot N$.

3. Perfectness of F/N

Lemma 3.1. *For any $g \in F_0$ and $h \in R^*$, the set $pr(R^1 \cap h.C(g))$ is dense in $\Pi_\Lambda D_v^1$ $\Pi_S SL_2(\mathcal{O}_v)$; therefore $F \cap h.C(g) \neq \varnothing$.*

Proof. By the definitions of F and F_0, for any $g \in F_0$, there exists x in Φ such that $gx \in$ Thus, we may assume that $g \in F$. Suppose now that L is any subfield of D, $L \neq$ then $L \cap N$ contains a generator u of L. Evidently we can find a sequence $\{n_r\} \in$ such that $gn_r \to u$ in D_v for every $v \in \Lambda \cup S$. It follows that any $z \in \Pi_{\Lambda \cup S} L_v^*$ is the lim of a sequence $z_r \in K(gn_r)$. Since L is arbitrary, we conclude that the group generat by $\{K(gn)^*; n \in N\}$ is dense in $\Pi_{\Lambda \cup S} D_v^*$

Choose $c \in C(g)$ with $\delta(c) = \delta(h)$. Since the commutator of

$$\Pi_\Lambda D_v^* \times \Pi_S GL_2(\mathcal{O}_v)$$

is

$$\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v)$$

it follows that

$$pr(C(g)) \bigcap (\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v))$$

is dense in

$$\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v).$$

Consider any element α in $\Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v)$. There exists a sequence $\{c_n\}$ in $C(g)$ such that

$$\{pr(c_n)\} \in \Pi_\Lambda D_v^1 \times \Pi_S SL_2(\mathcal{O}_v)$$

and $\{pr(c_n)\}$ converges to $pr(ch^{-1})\alpha$. Then the sequence $\{hc^{-1}c_n\}$ in $hC(g) \cap R^1$ satisfies $\{pr(hc^{-1}c_n\} \to \alpha$. This proves the first statement; the second follows obviously.

Proposition 3.2. $[F/N, F/N] = F/N$.

Proof. It suffices to show $F_U \subseteq [F,F]N$ (where $F_U = pr^{-1}(U) \cap R^1 \subseteq \tau(F_0)$): note that $F_U . N = F$.

Let $x \in F_U$. Then $x = \tau(f_0)$ for some $f_0 \in F_0$. By lemma 2.1, $\exists y \in R^*$ such that $\tau(f_0) = f_0 \ \sigma(f_0)^{-1} = f_0 y f_0^{-1} y^{-1} = [f_0, y]$. By lemma 3.1, $\exists f = yc_0 \in F \cap yC(f_0)$, so that $x = [f_0, y] \in [f_0, f]N$. Again $\exists f' = f_0 c' \in F \cap f_0 C(f)$ and so $x \in [f_0, f]N = [f', f]N$.

Lemma 3.3. *For any subalgebra* $L \neq K$ *of* D, $\delta(L) \cap \delta(F_0) = \delta(L \cap F_0)$.

Proof. That $\delta(L) \cap \delta(F_0) \supseteq \delta(L \cap F_0)$ is obvious.

Let $\delta(x) = \delta(y) \in \delta(L) \cap \delta(F_0)$.

Write $L_{\Lambda_s}^1$ for the group $\Pi_\Lambda L_v^1 \times \Pi_S (L_v \cap SL_2(\mathcal{O}_v))$.

We claim that $x^{-1} W_0 \cap L_{\Lambda_s}^1 \neq \emptyset$.

Now $pr(y) \in W_0 = W.\Phi$, so $y = w_v \phi_v$ in $D_v \forall v \in \Lambda \cup S$, where $\Pi_\Lambda w_v \times \Pi_S w_v \in W$, $\phi_v \in \Phi_v$. Since $\delta(\phi_v) = \delta(y) \forall v \in \Lambda \cup S$, therefore $x^{-1} \Pi_{\Lambda \cup S} \phi_v \in L_{\Lambda_s}^1 \cap x^{-1} W_0$. Choose any element $\Pi_\Lambda x_v \times \Pi_S x_v \in x^{-1} W_0 \cap L_{\Lambda_s}^1$. Since $pr(L^1)$ is dense in $L_{\Lambda_s}^1$, $\exists$ a sequence $\{z_n\} \in L^1$ such that $\{pr(xz_n)\} \to \Pi_\Lambda x_v \times \Pi_S x_v$. Writing $\Pi x_v = x^{-1}. \Pi y_v$ with $\Pi y_v \in W_0$, we have $\{pr(xz_n)\} \to \Pi y_v \in W_0$. Since W_0 is open, for some z_{n0} for big enough n_0, we have $pr(xz_{n_0}) \in W_0$. Thus $xz_{n_0} \in L \cap F_0$ and $\delta(xz_{n_0}) = \delta(x)$. Hence $\delta(L \cap F_0) = \delta(L) \cap \delta(F_0)$.

DEFINITION

Let $\hat{N}$ denote the kernel of the action of R^* on F/N.

that $\delta(F_0)=\delta(B)\cdot\delta(\hat{N}\cap F_0)$. By Corollary 1.6, $\exists n_1, n_2, \ldots n_{q+1}$ such that gn_i is nc
K and

$$B\subseteq(K(gn_i)\cap R^*)\cdot(K(gn_j)\cap R^*)\forall 1\leqslant i<j\leqslant q+1.$$

By the definition of q, $\exists t<s\leqslant q+1$ such that the images of $\delta(K(gn_t)\cap R^*)$
$\delta(K(gn_s)\cap R^*)$ in $\delta(R^*)/\delta(\hat{N}\cap F_0)$ coincide.

Write n in place of n_t. Hence the image of $\delta(B)$ in $\delta(R^*)/\delta(\hat{N}\cap F_0)$ is containe
that of $\delta(K(gn)\cap R^*)$. Thus,

$$\delta(B)\subseteq\delta(K(gn)\cap R^*)\cdot\delta(\hat{N}\cap F_0)$$

and so

$$\delta(F_0)=\delta(B)\cdot\delta(\hat{N}\cap F_0)\subseteq\delta(K(gn)\cap R^*)\cdot\delta(\hat{N}\cap F_0)$$

By lemma 3.3, therefore

$$\delta(F_0)=\delta(K(gn\cap F_0)\cdot\delta(\hat{N}\cap F_0)$$

Since $F_0\cap R^1=F$, it follows that

$$F_0=(K(gn)\cap F_0)\cdot(\hat{N}\cap F_0)\cdot F.$$

4. Proof of $F=N$

Suppose $F\neq N$. Let $Q=F_0/(\hat{N}\cap F_0)$, and let $p:F_0\to Q$ be the natural homomorphi
Without loss of generality, we can assume that F/N is simple (by replacing N by
radical of F/N, if necessary). Writing $G=p(F)$, we note then that $G\simeq F/N$. N
$Q^2\subseteq G$ since $f_0^2=\tau(f_0)$ mod $\hat{N}\cap F_0\forall f_0\in F_0$, and since $\tau(F_0)\subseteq F$. Also, $G\subseteq Q^2$ s

$$f=f_Un=\tau(f_0)n=f_0^2\alpha$$

mod $\hat{N}\cap F_0$ for some $\alpha\in\delta(R^*)$ and since $\delta(R^*)\subseteq\hat{N}\cap F_0$. Thus $Q^2=G$. Defin
$Q\to Q$ by $\phi(q)=q^2$. For $q\in Q$, choose $g\in F_0$ with $p(g)=q$.

By lemma 3.4, $Q=L_q\cdot G$ where $L_q=p(K(gn)\cap F_0)$.

Now (a) Q is finite since $F/N=G$.

(b) L_q is Abelian, and $q\in L_q\forall_q\in Q$. Thus $\phi(q\cdot\mathrm{Inv}(L_q))=\{\phi(q)\}$, where $\mathrm{Inv}(A)=\{x$
$x^2=e\}$ for any group A.

(c) For any finite Abelian groups $A\subseteq B$, $|\mathrm{Inv}(B)|\geqslant|\mathrm{Inv}(B/A)|$ where $|A|$ den
cardinality. So, by (b),

$$|\phi^{-1}(\phi(q))|\geqslant|\mathrm{Inv}(L_q)|$$

which is

$$\geqslant|\mathrm{Inv}(L_q/(L_q\cap G))|=|\mathrm{Inv}(Q/G)|=|(Q/G)|.$$

Since $|Q|=\Sigma_{g\in G}|\phi^{-1}(g)|$, if follows that

$Q = \mathrm{Inv}(Q) \times G$. Hence G cannot contain any element of order two. But G has ents of even order since every element of G is conjugate to its inverse [For, if then by lemma 2.1, $\exists y \in R^*$ with

$$yfy^{-1} = \sigma(f) = f^{-1}$$

by lemma 3.1, $F \cap y \cdot C(f) \neq$ and so $\exists u \in C(f)$ such that $yu \in F$. Then

$$p(f)^{-1} = p(yu)p(f)p(yu)^{-1}.$$

G is trivial, that is, $F = N$.

pter II: The general case

atements of the results

Global field of characteristic $\neq 2$

An absolutely almost simple, simply-connected K-anisotropic algebraic K-group e of the following types:

, (ii) A form of ${}^2A_l (l \geqslant 3)$ which splits over a quadratic extension of K, $_l(l \geqslant 2)$, (iv) $C_l (l \geqslant 2)$, (v) ${}^1D_l, {}^2D_l (l \geqslant 4)$.

ny finite set T of places of K, we call *T-adic* topology, the topology on $G(K)$ ed by its diagonal embedding $G(K) \to \Pi_T G(K_v)$. In particular, for $T = \varnothing$, this ogy is the trivial topology.

prove:

rem 5.1. *Let G be as above and assume that if G is a form of ${}^2A_3$ then it is isotropic ry finite completion. Let $T^* = \{v : v$ is a finite place of K and $G(K_v)$ is compact$\}$. en, any non-central normal subgroup of $G(K)$ is T-adically closed. In particular, for e above G except forms of A_3 which remain anisotropic over some finite etion, we have $T = \varnothing$, and thus $G(K)/Center$ is simple as an abstract group.*

is immediately yields the

OLLARY 5.2. (*Platonov's conjecture*)

as in the above Theorem, $G(K)$ is projectively simple $\Leftrightarrow$ *$G(K_v)$ is projectively for all finite places v of K.*

fact, our aim is to prove a more general result which is somewhat related to the uence subgroup problem for these groups. We proceed to state this result now. finite, non-empty set S of finite places of K containing the set T of all those v for which $G(K_v)$ is compact. Fix, $\forall v \in S$, an open compact subgroup Γ_v of $G(K_v)$. t

$$\Gamma := G(K) \cap \Pi_S \Gamma_v.$$

Note that Γ is an open (and hence closed) subgroup of $G(K)$ for the *S-adic* topol

The more general result alluded to, is:

Theorem 5.3. *Let G be as in Theorem 5.1. Then, any non-central normal subg N of Γ is open for the S-adic topology.*

Theorem 5.3 for groups of type A_1 has been established (Theorem 0.2) in Chapt We will use this in our proof of Theorems 5.1 and 5.3 above. For the form of ${}^2A_3$ which has been left out in Theorem 5.3, we can only prove:

Theorem 5.3′. *Let G be a form of type ${}^2A_3$ which remains anisotropic over some f completion of K. Then, we have the following implication: If $[G(K), G(K)]$ is T-adic open in $G(K)$, then any non-central normal subgroup of Γ is S-adically open.*

6. Equivalence of 'nearby forms' over local fields

Lemma 6.1. *Let D be a central division algebra of degree $\leqslant 2$ over a local field characteristic $\neq 2$ with an involution σ trivial on k. Let $n \geqslant 1$ be an integer ar denote the vector space of all σ-Hermitian forms on D^n. We can identify* S $\{(a_{i,j}) \in M_n(D)\colon a_{ij} = -\sigma(a_{j,i})\}$.

Let $h \in \mathsf{S}$ be non-degenerate. Then there is a neighbourhood Ω of h in S ar continuous map $\phi\colon \Omega \to GL_n(D)$ such that $\phi(h) = 1$ and for all $h' \in \Omega$, $h'(v, w) = h(\phi($ $\phi(h')w)$. In particular, h and h' are equivalent.

Proof. Consider the map $\alpha\colon M_n(D) \to \mathsf{S}$ given by $A \mapsto \theta(A)hA$, where $\theta = {}^t\sigma\colon M_n($ $M_n(D)$ is the map $(a_{ij}) \mapsto \sigma(a_{ji})$.

We see that $\theta(X) = -X, \forall X \in S$.

We note also that $\theta(AB) = \theta(B)\theta(A)$. The derivative map of α at the identity $M_n(D) \to M_n(D)$, given by $X \mapsto \theta(X)h + hX = hX - \theta(hX)$, (the coefficient of $\theta(1 + tX)\cdot h\cdot(1 + tX) = \theta(X)h + hX$). This is onto S, since $X \mapsto hX$ is an isomorph of $M_n(D)$ onto itself, and $B \mapsto B - \theta(B)$ is a map of $M_n(D)$ onto S, the last bei surjection due to the fact that $\theta^2 =$ identity and $M_n(D) = \mathsf{H} + \mathsf{S}$ is the eigen-sp decomposition of $M_n(D)$ with respect to θ, where $\mathsf{H} = \{X \in m_n(D)\colon \theta(X) = X\}$. T by the inverse function theorem over local fields, it follows that any h' in S clos h, is actually in its orbit and that $h' = \theta(A)hA$ for some A close to identity, in M

Remark 6.2. (1) When $D = k$, S is the space of quadratic forms on k^n so that lemma applies to quadratic forms.

(2) Let σ denote an involution on the quaternion division algebra D. Let A de the space of σ-anti Hermitian forms on D^n. Let $f_0 \in A$ be a non-degenerate form D^n. Let f_0 also denote the corresponding anti Hermitian matrix in $GL_n(D)$. T $x \mapsto \tau(x) = f_0\sigma(x)f_0^{-1}$ is an involution on $M_n(D)$ and the map $f \mapsto f_0 \cdot f$ is isomorphism of the vector space of σ-anti Hermitian forms onto the vector spac τ-Hermitian forms on D_n. Moreover it is easy to check that this isomorphism compatible with the action of $GL_n(D)$ on the two spaces. Thus the lemma 6.1 holds for σ-anti Hermitian forms.

7. 'T-adically' near groups of type A_1

First we make some definitions and remarks.

Let K denote a global field with $Char.(K) \neq 2$, and let G denote an absolute almost simple, simply-connected K-anisotropic algebraic K-group of one of th following types: (i) A_3, (ii) A form of type ${}^2A_l(l \geqslant 3)$ which splits over a quadrat extension of K, (iii) $B_l(l \geqslant 2)$, (iv) $C_l(l \geqslant 2)$, (v) ${}^1D_l, {}^2D_l(l \geqslant 4)$. Let S be a finite set finite places of K containing all those places v such that G is K_v-anisotropic (o equivalently $G(K_v)$ is compact). Fix, $\forall v \in S$, an open compact subgroup Γ_v of $G(K_v$ Let $\Gamma := G(K) \cap \Pi_S \Gamma_v$. (In case $S = \varnothing, \Gamma = G(K)$).

Remark 7.1. (I).. Any non-central normal subgroup N of Γ is of finite index in it.

To see this let S_0 be a finite set of places in the complement of S with S_0 containin all the Archimedean places. Let Φ be an S_0-congruence subgroup of G such th $\Phi \subseteq \Gamma$. Then, according to a theorem of Margulis [6], $N \cap \Phi$ has finite index in Now $N\Phi$ is an open and closed subgroup of $G(K)$ for the topology on $G(K$ induce by the diagonal embedding of $G(K)$ in the S_0-adele group $G(A_{S_0}(K))$. Now the closu of N in this topology in $G(K)$ is a normal subgroup of Γ and is hence easily seen be an open and closed subgroup of Γ. It follows that $N\Phi$ is the closure of N if Φ sufficiently small. Also, $\Gamma/N\Phi$ is finite—this group is naturally isomorphic to $\Gamma^\sim/N\Phi$ where $\Gamma^\sim$(resp. $N\Phi^\sim$) is the closure of Γ(resp. $N\Phi$) in $G(A_{S_0}(K))$. As $N\Phi/N \cong \Phi/\Phi \cap$ is finite, we conclude that Γ/N is finite.

(II).. For v outside S, *N is dense in $G(K_v)$*, since the closure of N in $G(K_v)$ is infinite normal subgroup of the projectively simple group $G(K_v)$. Moreover, the closu of N in $\Pi_S \Gamma_v$ is an open normal subgroup of finite index. We denote by $N^\wedge$, th S-adic closure of N in $G(K)$. Therefore, if $S = \varnothing$, we have $N^\wedge = G(K)$.

(III).. The groups G we are treating admit of the following description (Tits [20]): L D be a central algebra of degree $\leqslant 2$, V a vector space over D of dimension $\geqslant 3$ deg.$(D) = 2$ and $\geqslant 5$ if deg.$(D) = 1$. Let σ be an involution on D trivial on K and h non-degenerate, σ-Hermitian form on V. Then G is the simply-connected coverin group of $SU(h)$ (this description covers the groups of type A_3, B_n, C_n or D_n). Th groups of type B_n are covered by the $D = K$. The groups of type C_n are covered the case when deg.$(D) = 2$ and $D^\sigma = K$ while the case deg.$(D) = 2$, dim.$D^\sigma = 3$ cove the cases of A_3, ${}^1D_n$, ${}^2D_n$. The description in the case of groups of type A_n under o consideration is as follows. Here we take a quadratic extension D over K and G the group $SU(h)$. Note that the group $SU(h)$ is itself simply-connected except in th case of groups of type B_n and D_n. In the last two cases G is a central extension $SU(h)$ by μ_2 and is denoted Spin(h). We write $G(V)$ for the subgroup of $SU(V)($ of elements of spinor norm 1 in this case; we see that Spin$(V)(K)/\{1, -1\} \cong G(V)$

Construction 7.2

In the following, we fix:

and which are *T-adically* close to H in the following sense: $\forall v\in T, \exists\theta_n\in SU(V_v)$ ... $H_n(K_v)=\theta_n H(K_v)\theta_n^{-1}$ and $\theta_n\to 1$. Fix, $\forall v\in T$, a compact open subgroup Γ_v of ... and a fundamental system of compact open normal subgroups $\{W_n^v\}_n$ of Γ_v. ... $1\leqslant i\leqslant l$ be basis of V over D such that $h(e_i,e_j)=0$ if $i\neq j$. (D will be K, a qua... extension of K or a degree two division algebra over K depending on the gro... question). Let $r=2$ if $(D{:}K)=4$, and in all other cases, set $r=4$. Let W be the D... of e_i, $1\leqslant i\leqslant r$. Let h_W be the restriction of h to W. Note that h_W is a non-dege... σ-Hermitian form on W. Our assumptions on r guarantee that $SU(h_W)$ is a conn... semi-simple group. Let H_W denote the simply-connected covering group of SU... Let $\rho_n\in H_W(K)$ be any sequence of elements converging *T-adically* to the identit... that W is spanned as a D-vector space by e_1 and $\rho_n e_1$(resp. $e_1, e_2, \rho_n e_1, \rho_n e_2$) if... (resp. $r=4$). We fix any $g\in N^\wedge$. The set gN is dense in $N^\wedge$ for the *T-adic* top... So, it follows that $\exists x_n\in N$ such that $\forall v\in T$, gx_n is as close to ρ_n in $G(K_v)$ as we p... Fix, $\forall v\in T$, a maximal order $\Theta_v\subseteq D_v:=D\otimes K_v$ and a two-sided maximal ideal P_v ... We note that $\exists$ a sequence $\{a_n\}\to\infty$ such that

$$\forall v\in T,\ W_n^v\subseteq\{x\in\Gamma_v\colon\ x\cdot e_i-e_i\in\Sigma_i e_i\cdot \mathrm{P}_v^{a_n}\forall i\}.$$

Case I: $r=2$

We can assume $x_n\in N$ so chosen that, in the orthogonal decompositions

$$\rho_n\cdot e_1=e_1\cdot\alpha_n\perp e_2\cdot\beta_n$$

$$gx_n\cdot e_1=e_1\cdot\alpha_n'\perp v_n',$$

we have, $\forall v\in T$ that,

$$\beta_n\in D_v^* \text{ and } v_n'-e_2\cdot\beta_n\in e_2\cdot \mathrm{P}_v^{b_n}+\ldots+e_l\cdot \mathrm{P}_v^{b_n}$$

with $b_n-c_n^v\to\infty$ where $\beta_n\in \mathrm{P}_v^{c_n^v}\backslash \mathrm{P}_v^{c_n^v+1}$. We point out that our choice of x_n... exactly so as to make the vectors $v_n'\cdot\beta_n^{-1}-e_2\to 0$ in $V_u\forall v\in T$. Since the ... decompositions are orthogonal, we must have $e_1\cdot\alpha_n'$ close to $e_1\cdot\alpha_n$ and v_n' clo... $e_2\cdot\beta_n$. Call $v_n=v_n'\cdot\beta_n^{-1}$ and note that $v_n\to e_2$ in $V_v\forall v\in T$. Write $V_n=e_1D\perp v_nD$... H_n for the subgroup $G(V_n)$ of $G(V)$.

Case II: $r=4$

Get a sequence $x_n\in N$ such that, on writing

$$\rho_n\cdot e_1=\perp_{1\leqslant i\leqslant 4}e_i\alpha_n^i$$

$$gx_n\cdot e_1=e_1\cdot\phi_n\perp u_n'$$

we have, $\forall v\in T$, $u_n'/\alpha_r^3-e_3\to 0$ in V_v and

$$\rho_n\cdot e_2\perp_{1\leqslant i\leqslant 4}e_i\cdot\beta_n^i$$

Given $g \in N^{\wedge}$, $T \supseteq S$, our choice of W, $V_n \subseteq V$, H, H_n satisfies: $\forall v \in T$, $\exists \theta_n^v \in SU(V_v)$ such that $H_n(K_v) = \theta_n^v \cdot H(K_v) \cdot (\theta_n^v)^{-1}$ for $n \gg 0$ and $\theta_n^v \to 1$ in $SU(V_v)$.

Proof. In all the cases, $\forall v \in T$, the restrictions of h to the submodules W_v and V_{nv} of V_v are arbitrarily close in the vector space of σ-Hermitian forms over D_v. By the lemma in the last section, $\forall v \in T$, W_v and V_{nv} are isometric by elements close to the identity in $M_r(D_v)$. Note that the lemma mentioned also includes the situation when $D_v \cong M_2(K_v)$. By Witt's theorem, valid for general α-Hermitian spaces [S], we can extend these to isometries of V_v and hence get $\forall v \in T$, θ_n as claimed.

8. Bound for 'congruence' levels of H_n

Lemma 8.1. *Let k be a non-Archimedean, non-dyadic local field of characteristic $\neq 2$, f its residue field and let D be the quaternion division algebra over k. Write Θ, P for the maximal compact subring of integers of D and its unique maximal two-sided ideals respectively.*

Let D^1 denote $\{x \in D^*: N_{red}(x) = 1\}$, and $N_i = \{x \in D^1: x \equiv 1 \bmod \mathsf{P}^i\} \forall i \geqslant 1$ (a) Given any $r > 0$, $\exists m$ such that, for each non-central normal subgroup $N \subseteq D^1$ of index $\leqslant r$, we have $N \supseteq N_m$. (b) If $N \subseteq D^1$ is a non-central subgroup such that $|D^1 : N| < (|f| + 1)/2$, then $[D^1, D^1] \subseteq N$.

Proof. For $i \geqslant 1$, let G_i denote the subgroup $\{x \in D^1 : x \equiv \text{some unit in } k \bmod \mathsf{P}^i\}$; it is also convenient to write $G_0 = D^1$. Now, by [15], any non-central normal subgroup N has a 'level' i.e. $\exists$ a maximal i with $G_i \supseteq N$. Thus, bounding the index bounds the 'level'. Moreover [15] says that, if N has level i, then, $N_{i+1} \subseteq N \subseteq G_i$. This proves (a). Also, since $[D^1, D^1] = N_1$, we have $|D^1 : N_1| = |f| + 1$. Since $|G_1 : N_1| = 2$, if $|D^1 : N| < (|f| + 1)/2$, we must have 'level' of $N = 0$, and hence $N \supseteq N_1 = [D^1, D^1]$.

Notations. Let $N \subseteq \Gamma$ be a non-central normal subgroup. Take any $g \in N^{\wedge}$. Define $S_0 = \{v: v \text{ finite and } |f_v| \leqslant 2r\}$, where f_v stands for the residue field, and $r = |\Gamma/N|$. Let $S_1 = \{v: v(2) \neq 0\}$, and suppose $T \supseteq S \cup S_0 \cup S_1$ is a non-empty, finte set of finite places. Given this g and T consider the groups H_n and elements θ_n^v as in Construction 7.2.

Let $S_n = \{v : v \text{ is a finite place of } K, \text{ and } H_n(K_v) \text{ is compact}\}$.

PROPOSITION 8.2 $\exists m$*(independent of* n*) such that*

$$N \cap H_n(K) \supseteq H_n(K) \cap \prod_T (H_n(K_v) \cap W_m^v) \cdot \prod_{S_n \setminus T} [H_n(K_v), H_n(K_v)]$$

Proof. Let Ψ_n denote the closure of $N \cap H_n(K)$ in $\Pi_{S \cup S_n} H_n(K_v)$. Then by the main (Theorem 2) of Chapter I, $N \cap H_n(K) = \Psi_n \cap H_n(K)$. Now $|\Gamma \cap H_n(K): N \cap H_n(K)| \leqslant |\Gamma: N| = r$. So, $|\Pi_{S_n \setminus S} H_n(K_v) \times \Pi_S (H_n(K_v) \cap \Gamma_v): \Psi_n| \leqslant r$.

Thus,

$$|H_n(K_v): \Psi_n \cap H_n(K_v)| \leqslant r \forall v \in S_n \setminus S$$

and

$$|H_n(K_v)\cap\Gamma_v\colon \Psi_n\cap H_n(K_v)|\leqslant r\,\forall v\in S.$$

Firstly, consider any $v\in S_n\setminus T$.

Now, $\Psi_n\cap H_n(K_v)$ are open normal subgroups of $H_n(K_v)$, of index $\leqslant r<|f_v|/2$ (sin $v\in S_n\setminus T$). Consequently, for $v\in S_n\setminus T$, $H_n(K_v)\cap\Psi_n\supseteq[H_n(K_v),H_n(K_v)]$. Next let $v\in(T$ $S_n)\setminus S$. Note that, then $H(K_v)$ is compact. Moreover, $\forall n$, $(\theta_n^v)^{-1}\cdot(H_n(K_v)\cap\Psi_n)\cdot\theta_n^v$ a subgroups of $H(K_v)$ of index $\leqslant r$. By Lemma 8.1 (a), $\exists m$ (independently of n) su that, for $n\gg 0$, $(\theta_n^v)^{-1}\cdot(H_n(K_v)\cap\Psi_n)\cdot\theta_n^v\supseteq H(K_v)\cap W_m^v$. As $\theta_n^v\to 1$ we have $H_n(K_v)$ $\Psi_n\supseteq H_n(K_v)\cap W_m^v$ for $n\gg 0$. Clearly, we can choose an m which is common for the places $(T\cap S_n)\setminus S$. Now, consider any $v\in S$. $\forall n$, $(\theta_n^v)^{-1}\cdot(H_n(K_v)\cap\Psi_n\cdot\theta_n^v$ are subgrou of $H(K_v)\cap\Gamma_v$ of index $\leqslant r$. If $H(K_v)$ is compact, then, again we are done by t Lemma 8.1 (a).

9. Centrality of $N^\wedge/N$

Consider (as in the last section) G of type A_3, ${}^1D_l$, ${}^2D_l(l\geqslant 4)$, or $B_l(l\geqslant 2)$. Suppose containing $\{v$ finite: $G(K_v)$ is compact$\}$ is a finite set consisting of finite places. L $\Gamma=G(K)\cap\Pi_S\Gamma_v$, where Γ_v; $v\in S$ are open compact subgroups. In particular, $\Gamma=G(I$ if $S=\varnothing$. Let $N\subseteq\Gamma$ be a non-central normal subgroup and let $N^\wedge$ denote its *S-ad* closure in $G(K)$. Recall that $G(K)=\mathrm{Spin}(V,h)\,(K)$ and we write $G(V)$ for the grou $\mathrm{Spin}(V)(K)/\{1,-1\}\cong$ the subgroup of $SU(V)(K)$ of spinor norm 1 elements. W assume that N is normalized by a subgroup $\Gamma^\sim$ of $SU(K)$ of the form

$$\Gamma^\sim=SU(K)\cap\prod_S\Gamma_v^\sim$$

where $\Gamma_v^\sim$ is a compact open subgroup of $SU(K_v)$ normalizing Γ_v. There is no lo of generality in making this assumption. In fact we see that Γ is normal in $\Gamma^\sim$ ar if $N\subseteq\Gamma$ is a normal subgroup of index r, then $N_0=$ group generated by $\{x^r\colon x\in$ is normalized by $\Gamma^\sim$ and is contained in N and we replace N by N_0. In the sequ then we assume that N is normal in $\Gamma^\sim$. We also note that in the case of $\mathrm{Spin}(V,$ for a skew-Hermitian space over a quaternion algebra over K, it is known by [1 Th. 7.1] that $U(V,h)=SU(V,h)$. We also fix $\forall v\in S$, a sequence $\{W_n^v\}_n$ of open compa subgroups of Γ_v which contract to the identity.

Theorem 9.1. *For G of type A_3, ${}^1D_l$, ${}^2D_l(l\geqslant 4)$, $[SU(X)\cap\Gamma^\sim,N^\wedge]\subseteq N$, $\forall X\subseteq V$ dimension 1. For G of type $B_l(l\geqslant 2)$, $[SO(X)\cap\Gamma^\sim,N^\wedge]\subseteq N$, $\forall X\subseteq V$ of dimension In particular, when $\Gamma=G(K)$, or equivalently $S=T$, we have $[G(K),N^\wedge]\subseteq N$.*

Note that $G(X)$ is a K-anisotropic torus splitting over a quadratic extension $K(X$ as in the statement of the theorem).

Lemma 9.2. Consider the case when G is A_3 or D_l. Let $e\in V$, $e\neq 0$. Then

Proof. Let $e \in X \subseteq V$ with $\dim X = 2$. Then, $N \cap G(X) \supseteq G(X) \cap \Pi_{S_X}(G(X_v) \cap W_r^v)$ for some $r > 0$. Here $S_X = \{v \text{ finite: } G(X_v) \text{ is compact}\}$. So,

$$N \cap G(eD) \supseteq G(eD) \cap \prod_{S_X} (G(eD_v) \cap W_r^v).$$

Consider any $v \in S_X \backslash S$. Since $G(V_v)$ is isotropic, $\exists f_v \in V_v$ such that $G(eD_v + fD_v)$ is isotropic. By weak approximation, we get $f \in V$ such that $G(eD_v + fD_v)$ is isotropic $\forall v \in S_X \backslash S$ (since nearby forms are equivalent).

Letting $X^{\sim} = eD + fD$, we have $S_X \cap S_{X^{\sim}} \subseteq S$, where $S_{X^{\sim}} = \{v \text{ finite: } G(X^{\sim}_v \text{ is compact}\}$. Thus we have

$$N \cap G(eD) \supseteq G(eD) \cap \prod_{S_{X^{\sim}}} (G(eD_v) \cap W_s^v)$$

for some $s > 0$. Using weak approximation for the torus $G(eD)$, we will get

$$N \cap G(eD) \supseteq G(eD) \cap \prod_{S} (G(eD_v) \cap W_t^v)$$

for some $t > 0$. In fact, if $\alpha \in G(eD) \cap \Pi_S(G(eD_v) \cap W_t^v)$, then we get $\alpha_n \in G(eD) \cap \Pi_{S_X}(G(eD_v) \cap W_r^v)$ such that $\alpha_n \to \alpha$ in $\Pi_{S_X \backslash S} G(eD_v)$. Then, for $n \gg 0$,

$$\alpha_n^{-1} \alpha \in G(eD) \cap \prod_{S_{X^{\sim}}} (G(eD_v) \cap W_s^v) \subseteq N \cap G(eD).$$

Thus, $\alpha \in N \cap G(eD)$, since $\alpha_n \in N \cap G(eD)$.

Exactly similarly we have:

Lemma 9.3. *Consider the case when G is of type $B_l (l \geqslant 2)$. Let $X \subseteq V$ with $\dim \cdot X = 2$. Then,*

$$N \cap G(X) \supseteq G(X) \cap \prod_{S} (G(X_v) \cap W_t^v)$$

for some $t > 0$.

Remark 9.4 (Morita equivalence). (i) ([18], P. 361–362) Consider an arbitrary field L of characteristic $\neq 2$ and let $A = M(2, L)$ with the canonical involution

$$\sigma: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Let V be a free A-module of rank n and $h: V \times V \to A$ a skew-Hermitian form on V. For

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

for the 1-dimensional form $h = \langle a \rangle$ i.e. for

$$h(x,y) = x^{\sigma} a y, \quad a = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

the form b has the matrix

$$\begin{pmatrix} \gamma & -\alpha \\ -\alpha & -\beta \end{pmatrix}$$

with respect to the basis e_1, e_2. In particular, det.h = det.b, and by diagonalisation this holds for arbitrary h. So, b is regular if, and only if, h is. Moreover h can be reconstructed from b as

$$h(x,y) = \begin{pmatrix} -b(xee_1, ye_1) & -b(xee_1, yee_1) \\ b(xe_1, ye_1) & b(xe_1, yee_1) \end{pmatrix}$$

where

$$e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

thus, h is completely determined by b and $h \cong h' \Leftrightarrow b \cong b'$. h and b are said to be Morita equivalent. h is isotropic if b has Witt's index $\geqslant 2$.
(ii) By ([18], Thms. 10.3.6, 10.3.7), over the quaternion division algebra over a local field k, a skew-Hermitian space (X, h) of dimension $\geqslant 2$ is anisotropic if, and only if, k is non-Archimedean and either dim.$X = 2$, disc.$X \neq 1$, or dim.$X = 3$, disc.$X = 1$.

DEFINITIONS

On a skew-Hermitian space (V, h), a quasi-reflection $\tau_{x,\alpha}$ for $x \in V, \alpha \in D$ is defined by

$$\tau_{x,\alpha}(y) = y - x.\alpha h(x,y), \quad y \in V.$$

It is in $SU(V,h)$ when $\alpha^{-1} - (\alpha^{-1})^{\sigma} = h(x)$.

Lemma 9.5. *Let (X, h) be a 2-dim. anisotropic, skew-Hermitian space over a quaternion division algebra D over K. Let v be a finite place such that $D \otimes_K K_v$ splits and $SU(X_v, h_v)$ is anisotropic. Let $e, f \in X$ and $x \in SU(eD)$, $y \in SU(fD)$. Then, $\exists a \in X$ such that $[x,y] \in G(aD_v)$ modulo $[G(X_v), G(X_v)]$.*

Proof. By (ii) of Remark 9.4 on Morita theory, we have

$$SU(X_v, h_v) \cong SO(q_v) \cong \{(a,b) \in A^* \times A^* : N(a) \cdot N(b) = 1\} / K_v^*$$

where A is the unique quaternion division algebra over K_v and q_v is its norm form; and K_v^* sits as $\{(t, t^{-1}) : t \in K_v^*\}$. Now, $SU(eD_v)$ and $SU(fD_v)$ correspond to certain quadratic extensions; indeed, we can

nical involution of A. It is known [15] that $[U, U] = [A^1, A^1]$. Using this identity the fact that $G(X_v) \cong A^1 \times A^1/\{\pm(1,1)\}$ we calculate $[x, y]$ modulo $[G(X_v),$ $)]$ and find that it is of the form (z, z^{-1}) for some $z \in A^1$. But, $\{(z, z^{-1}): z \in F^1\}$ quadratic extension $K_v \subseteq F \subseteq A$ is $\cong G(aD_v)$ for some $a \in X$.

f of Theorem 9.1. G of type A_3, ${}^1D_l$, ${}^2D_l (l \geqslant 4)$.

call that we can realize $G(K)$ as $\mathrm{Spin}(V, h)$, dim. $V \geqslant 3$ where (V, h) is a degenerate, anisotropic, σ-skew–Hermitian space over a quaternion division ra D over K.

e identify $G(V) = \mathrm{Spin}(V, h)/\{1, -1\}$ with $Ker(\theta)$, where θ: $SU(V, h) \to K^*/K^{*2}$ is pinor norm.

t $g \in N^\wedge$.

ke any $W = e_1 D \perp e_2 D \subseteq V$ and $x \in \Gamma \cap SU(e_1 D)$. Note that $\Gamma \cap SU(e_1 D) \neq \{1\}$ $SU(e_1 D)$ is a torus which splits over a cyclic (indeed a quadratic) extension of d consequently the weak approximation theorem is valid for it. We shall show $[x, g] \in N$.

fine $S_0 = \{v$ finite: $|f_v| \leqslant 2. |\Gamma/N|\}$, $S_2 = \{v: v(2) \neq 0\}$ $S_{\mathrm{ram}} = \{v$ finite: $D \otimes_K K_v$ is ision algebra$\}$. Let $T \supseteq S \cup S_0 \cup S_2 \cup S_{\mathrm{ram}}$ be a finite non-empty set consisting of places. We can get $x_n \in N$ such that the two-dimensional spaces $V_n = e_1 D + g x_n \cdot$ tend *T-adically* to W i.e. the groups $H_n = Spin(V_n, h)$ and $H = Spin(W, h)$ satisfy:

$$\forall v \in T, \quad \exists \theta_n \in SU(V_v), \quad \theta_n \to 1$$

that $H_n(K_v) = \theta_n . H(K_v) \cdot \theta_n^{-1}$. We can write $g x_n \cdot e_1 = h_n \cdot e_1$ for a quasi-reflection $U(V_n, h)$. In fact, $g x_n \cdot e_1 = \tau_{e_1 - g x_n \cdot e_1, \alpha_n} \cdot e_1$ where $\alpha_n = h(e_1 - g x_n \cdot e_1, e_1)^{-1}$. Note that $- g x_n \cdot e_1, e_1) \neq 0$ since $h(e_1 - g x_n \cdot e_1, e_1) - h(e_1 - g x_n \cdot e_1, e_1)^\sigma = h(e_1 - g x_n \cdot e_1, e_1 -$ $_1) \neq 0$. For any $v \in T$, we consider the orbit maps $U(V_n)(K_v) \to V_n(K_v)$; $g_v \to g_v \cdot e_1$. $g x_n \cdot e_1 \to e_1$ and $V_n(K_v) \to W_v$, we can apply the inverse function theorem rmly for all $n \gg 0$ and get $g x_n \cdot e_1 = \sigma_n^v \cdot e_1$ with $\sigma_n^v \in SU(V_n)(K_v)$, and $\sigma_n^v \to 1$ as o. Consider $v_n D$ where $V_n = e_1 D \perp v_n D$. Since $SU(v_n D)$ is a torus split by a cyclic ed a quadratic) extension of K, it satisfies the weak approximation property. efore, we can choose $h_n \in SU(V_n)$ such that $h_n \to 1$ in $SU(V_n)(K_v) \forall v \in T$, and $_1 = h_n \cdot e_1$. Since $x \in \Gamma \cap SU(e_1 D)$, we have $[x, g x_n] = [x, h_n]$ i.e. $[x, g] = [x, h_n]$ N, $\forall n$. Hence, it suffices to show that the elements $[x, h_n]$ in $H_n(K)$, belong to r $n \gg 0$. Recall from Proposition 8.2 that

$$N \cap H_n(K) \supseteq H_n(K) \cap \prod_T (H_n(K_v) \cap W_m^v) \cdot \prod_{S_n \setminus T} [H_n(K_v), H_n(K_v)].$$

nay assume that $S_n \setminus T \neq 0$ for otherwise we are done since $[x, h_n] \to 1$ *T-adically.* that $\forall v \in S_n \setminus T$, D_v is split and hence by Remark 9.4 (ii) we have $H_n(K_v) \cong A^1 \times A^1$ e A is the unique quaternion division algebra over K_v. Now, by lemma 9.5, we change $[x, h_n]$ by an element of $N \cap H_n(K)$ to get an element $y_n \in H_n(K)$ such that 1 *T-adically* and $y_n \in G(a_n D_v)$ for some $a_n \forall v \in S_n \setminus T$. Thus, $y_n \in G(a_n D)$, $y_n \to 1$ *ically.* By lemma 9.2, we have $y_n \in N$ for $n \gg 0$. Thus, $[x, g] \in N$. In case $\Gamma = G(K)$, vill have $[SU(K), N^\wedge] \subseteq N$, since $SU(V)$ is generated by quasi-reflections. This es that Theorem for G of the above types.

f of Theorem 9.1 *for G of type* $B_l (l \geqslant 2)$. We realize $G(K)$ as $Spin(V, h)$ where

identify $G(V) := Spin(V)/+_-1$ with $Ker(\theta)$, where $\theta: SO(V,h) \to K^*/K^{*2}$ is the s
norm.

Consider any $g \in N^\wedge$.

Let $X = e_1 K \perp e_2 K \subseteq W = \perp_{1 \leqslant i \leqslant 4} e_i K$ and $x \in SO(X) \cap \Gamma$.

We shall show that $[x, g] \in N$.

Once again, let $S_0 = \{v \text{ finite: } |f_v| \leqslant 2. |\Gamma/N|\}$, $S_2 = \{v: v(2) \neq 0\}$. Take $T \supseteq S \cup S$
to be a non-empty, finite set consisting of finite places of K. We can get, $x_n \in N$
that the spaces $V_n = e_1 K + e_2 K + g x_n \cdot e_1 K + g x_n \cdot e_2 K$ tend T-*adically* to W i.
groups $H_n = Spin(V_n, h)$ and $H = Spin(W, h)$ satisfy: $\forall v \in T$, $\exists \theta_n \in SO(V_v), \theta_n \to 1$
that $H_n(K_v) = \theta_n \cdot H(K_v) \cdot \theta_n^{-1}$.

By Witt's theorem, we can write, for some $h_n \in SO(V_n, h)$ that

$$g x_n \cdot \{e_1, e_2\} = h_n \cdot \{e_1, e_2\}$$

i.e. $g x_n \cdot e_1 = h_n \cdot e_1$ and $g x_n \cdot e_2 = h_n \cdot e_2$. But, $\forall v \in T$, we can apply the inverse fun
theorem to the family of subgroups $SO(V_n(K_v))$ and get $\sigma_n^v \in SO(V_{nv})$ with $\sigma_n^v -$
$n \to \infty$ such that $g x_n \{e_1, e_2\} = \sigma_n^v \{e_1, e_2\}$. Consider the torus $SO(u_n K \perp v_n K)$, v
$V_n = e_1 K \perp e_2 K \perp u_n K \perp v_n K$. Applying the weak approximation theorem (which
for it since it splits over a quadratic extension of K) at the places T, we get eler
$h_n \in SO(V_n)$ such that $h_n \to 1$ in $SO(V_n(K_v)) \forall v \in T$ and such that $g x_n \{e_1, e_2\} = h_n \{e_1$
So, $[x, g] = [x, g x_n] \bmod N = [x, h_n] \bmod N$. Consequently, it suffices to show
for large enough n, $[x, h_n] \in N$. This is seen exactly as in the last case, once we ob
that $H_n(K_v) \cong A^1 \times A^1 \forall v \in S_n$. Hence, $[x, g] \in N \forall g \in \hat{N}$ and $x \in SO(X) \cap \Gamma$.

10. Normal subgroups of Γ

Let G be one of the following: (i) ${}^1A_3$ or a form of ${}^2A_3$ which is isotropic ov
finite completions (ii) ${}^1D_l$, ${}^2D_l (l \geqslant 4)$, (iii) $B_l (l \geqslant 2)$, (iv) $C_l (l \geqslant 2)$, (v) a form of ${}^2A_l($
splitting over a quadratic extension of K.

Let $S \supseteq T = \{v \text{ finite: } G(K_v) \text{ is compact}\}$ be any finite set consisting of finite pl
Fix, $\forall v \in S$, open compact subgroups Γ_v of $G(K_v)$. Let $\Gamma = G(K) \cap \Pi_S \Gamma_v$. Recall
we call the topology inherited by $G(K)$ via the diagonal embedding $G(K) \to \Pi_S G$
the S-adic topology. Then, we prove:

Theorem 10.1. *Any non-central normal subgroup of* Γ *is S-adically closed.*

We note immediately the

COROLLARY 10.2. (*Platonov's conjecture*)

For G as above, G(K) is projectively simple $\Leftrightarrow$ *$G(K_v)$ is projectively simple for all*
places v.

For the groups of type ${}^2A_3$ which remain anisotropic over some finite comple
we can show only:

Theorem 10.3. *Let G of type ${}^2A_3$ remain anisotropic over some finite completic*
K. Then, non-central normal subgroups of Γ *are S-adically closed if* $[G(K), G(K$

Remark 10.4. Note that to prove the theorem, we may assume without loss generality that $\Gamma_v = G(K_v) \forall v \in T$. For, if $N \subseteq \Gamma$ is a non-central normal subgrou then $\exists N_0 \subseteq N$ of finite index which is normal in $\Gamma^{\sim} = G(K) \cap \Pi_{S \setminus T} \Gamma_v$ since $\Gamma_v \subseteq G(K$ is of finite index $\forall v \in T$. Knowing that N_0 is *S-adically* closed, it follows immediate that N is *S-adically* closed.

Notation. Let $N \subseteq \Gamma$ be a non-central normal subgroup and let $N^{\dagger}$ denote its *S-ad* closure. We first show that the proofs of Theorem 10.1 for the cases $C_l (l \geqslant 3)$ ar ${}^2A_l (l \geqslant 4)$ under our consideration follow from $C_2 = B_2$ and ${}^2A_3$.

Proof for G of type $C_l (l \geqslant 3)$. Realize $G(K)$ as $SU(V, h)$ where (V, h) is a non-degenera anisotropic Hermitian space of dimension $\geqslant 3$ over a quaternion division algebra over K. We shall apply induction on dim V. Let $g \in N^{\dagger}$. Let $e_1 \in W \subseteq V$ whe $\dim(W) = 2$. Get $x_n \in N$ such that $gx_n \to 1$ *S-adically* i.e. $W_n = e_1 D + gx_n e_1 D \to$ *S-adically*. Consider $H_n = SU(W_n)$. By Witt's theorem for Hermitian forms and we approximation for $SU((e_1 D)^{\perp})$, $gx_n e_1 = h_n e_1$ for some $h_n \in SU(W_n)$ with $h_n \to 1$ *S-adical* By assuming the result for C_2, we get by induction that since $(e_1 D)^{\perp}$ has dimensi $\geqslant 2$, that $h_n^{-1} gx_n, h_n \in N$ for $n \gg 0$, thus, $g \in N$.

Proof for G of type ${}^2A_l (l \geqslant 4)$ *split by a quadratic extension.* Realize $G(K)$ as $SU(V$ where (V, h) is a non-degenerate anisotropic Hermitian space of dimension $\geqslant 5$ ov a quadratic extension L of K. Let $g \in N^{\dagger}$. Let $e_1 \in W \subseteq V$ with $\dim(W) = 4$. Get $x_n \in$ such that $gx_n \to 1$ *S-adically* i.e. $W_n = e_1 L + e_2 L + e_3 L + gx_n e_1 L \to W = e_1 L + e_2 L$ $e_3 L + e_4 L +$ *S-adically*. Consider $H_n = SU(W_n)$. Again by Witt's theorem Hermitian spaces $gx_n e_1 = h_n e_1$ for some $h_n \in H_n$. We can also assume that h_n *S-adically* on using weak approximation for $SU((e_1 L)^{\perp})$. By induction hypothesis, will get $h_n^{-1} gx_n, h_n \in N$ for $n \gg 0$. So $g \in N$.

PROPOSITION 10.5.

As above, $T = \{v$ *finite place:* $G(K_v)$ *is compact*$\}$. *Let G be one of the types* A_3, ^{1} ${}^2D_l (l \geqslant 4)$, $B_l (l \geqslant 2)$. *If* $[G(K), G(K)]$ *is T-adically closed, then any non-central norm subgroup of* $G(K)$ *is T-adically closed.*

Proof. We consider the following two topologies on $G(K)$: The first is the one whe a fundamental system of neighbourhoods around the identity is given by the fam of normal subgroups of finite index. The second is the *T-adic* topology. We c 'complete' $G(K)$ with respect to these topologies and call the completions G^* and Note that $G^{\wedge} = \Pi_T G(K_v)$. We have an exact sequence

$$1 \to \mathsf{C} \to G^* \to G^{\wedge} \to 1 \ldots (*)$$

where $\mathsf{C} = \lim_{\leftarrow} N^{\wedge}/N$. According to Th. 9.1, $(*)$ is a central extension. It follows th the central extension $(*)$ splits, by [12], [13], since it does over $G(K)$. Consequent the triviality of C would follow from:

PROPOSITION 10.6.

Hom$(\mathsf{C}, S^1) \cong$ *Coker*$(H^1(G^{\wedge}, S^1) \to H^1(G^*, S^1)) \cong [G(K), G(K)]^{\wedge}/[G(K), G(K)]$. *He*

Proof. The Hochschild–Serre spectral sequence corresponding to $C \subset G^*$, with S^1-coefficients is given by $E_2^{p,q} = H^p(G^\wedge, H^q(C, S^1))$. $E_\infty^{0,1} = E_3^{0,1} = \mathrm{Ker}(E_2^{0,1} \to E_2^{2,0}) = \mathrm{Ker}(\mathrm{Hom}(C, S^1) \to H^2(G^\wedge, S^1))$. $E_\infty^{1,0} = E_2^{1,0} = H^1(G^\wedge, S^1)$. Hence, $0 \to E_\infty^{1,0} \to H^1(G^*, S^1) \to E_\infty^{0,1} \to 0$ gives Coker $(H^1(G^\wedge, S^1)) \to H^1(G^*, S^1) \cong E_\infty^{0,1}$. But $E_\infty^{0,1} = \mathrm{Hom}\,(C, S^1)$ since (*) is trivial. Thus, the first isomorphism in the theorem follows. We are interested in showing that $[G(K), G(K)]^\wedge/[G(K), G(K)] \cong \mathrm{Ker}(G^*/[G^*, G^*] \to G^\wedge/[G^\wedge, G^\wedge])$. Now, since $G(K)$ is dense in G^* and $[G^*, G^*]$ is open in G^*, $G(K)$. $[G^*, G^*] = G^*$. Thus, $G(K) \to G^*/[G^*, G^*]$ is surjective. Its kernel $G(K) \cap [G^*, G^*] = [G(K), G(K)]$, since $[G(K), G(K)]$ is open (in the arithmetic topology) as well as dense in $G(K) \cap [G^*, G^*]$. So, $G(K)/[G(K), G(K)] \cong G^*/[G^*, G^*]$. Therefore, Ker $(G^*/[G^*, G^*] \to G^\wedge/[G^\wedge, G^\wedge]) \cong G(K) \cap [G^\wedge, G^\wedge]/[G(K), G(K)] = [G(K), G(K)]^\wedge/[G(K), G(K)]$. This proves the proposition.

We will now use the above Proposition to prove the following (of which it is a special case).

PROPOSITION 10.7.

Let G be as in Prop. 10.5. Let $S \supseteq T$ be any finite set of non-Archimedean places. Assume that $[G(K), G(K)]$ is T-adically closed. Then, the extension

$$1 \to C \to G^* \to \hat{G} \to 1 \ldots (*)$$

where $\hat{G}$ (resp. G^*) is the completion of $G(K)$ with respect to S-adic closed subgroups of finite index (resp. all subgroups of finite index) is a central extension. Moreover, every non-central normal subgroup of Γ is S-adically closed.

Proof. Let $Z(C)$ be the centraliser of C. By Th. 9.1, $Z(C)$ is infinite normal in $G(K)$. Hence $Z(C)$ is *T-adically* closed and open in $G(K)$. In Th. 9.1, it was shown that $Z(C) \supseteq A =$ Group generated by $\{\Gamma \cap G(X)\}$ for various X of dimension 1 over D (or 2 over K) according as G is of type D_n (or type B_n). Now, by weak approximation at the places in S one knows that Γ is dense in $\Pi_T G(K_v) \times \Pi_{S-T} \Gamma_v$ and since $Z(C)$ is open in $\Pi_T G(K_v)$, one sees by projecting to this factor that $Z(C) = G(K)$. This proves that (*) is central. Exactly as before applying the Hochschild–Serre spectral sequence we conclude that C is trivial. Note that $[G(K), G(K)]$ being *T-adically* closed is, *a fortiori*, *S-adically* closed. So the proposition is proved.

We note that this Proposition proves Th. 10.3.

Hereafter, we proceed to prove:

Theorem 10.8. *Let G be as in Theorem 10.1. Let $S = \{v$ finite: $G(K_v)$ is compact$\}$, $N = [G(K), G(K)]$. Then, N is S-adically closed in $G(K)$.*

Proof for type ${}^1A_3$. It is known by [11], [14] that, in this case $[G(K), G(K)]$ is *S-adically* closed.

Proof for ${}^2A_3$, ${}^1D_l$, ${}^2D_l (l \geqslant 4)$. We can write $G(K) = \mathrm{Spin}(V, h)$, $\dim V \geqslant 3$, and $G(V) = \{g \in SU(V, h) : \theta(g) = 1\} = \mathrm{Spin}(V)/\pm 1$ where $\theta: SU(V, h) \to K^*/K^{*2}$ is the spinor norm map. We firstly prove the following simple:

is isotropic over all finite completions of K. *(ii) For* $\dim V=3$, $\exists W\subseteq V$, $\dim W=2$ *such that* (W_v, h_v) *is totally isotropic for almost all places* v *of* K.

Proof of Lemma. To prove (i), let $e_1, e_2 \in V$ be such that $\dim(e_1D+e_2D)=2$. The group $SU(e_1D+e_2D)$ is locally isotropic at almost all completions of K, say all finite completions except a finite set A. Since $SU(V,h)$ is isotropic at all finite places, we can get $e_v, v\in A$ such that $SU(e_1D_v+e_2D_v+e_vD_v)$ is isotropic. By weak approximation at the places A, we can get $e_3\in V$ such that $SU(e_1D+e_2D+e_3D)$ is isotropic at all finite completions since, nearby forms over a local field are isomorphic, by lemma 6.1.

To prove (ii), it is enough to find $W\subseteq V$ of dimensions 2 such that $\operatorname{disc}(W,h)=1$. For, at all finite places v where $D_v\cong M_2(K_v)$, the group $SU(W_v,h_v)\cong SO(q_v)$, where q_v is a 4-dimensional quadratic form over K_v, of discriminant $=\operatorname{disc}(W_v,h_v)=1$, and since, for almost all v, q_v has to be isotropic, so $q_v=H\perp H$. To find such a W is to find a vector $e\in V$ such that $N_{\text{red}}h(e)=\operatorname{disc}(V,h)$ in K^*/K^{*2}. Firstly, we find a $\lambda\in D^*$ such that $\lambda^\sigma=-\lambda$ and $N_{\text{red}}\lambda=d:=\operatorname{disc}(V,h)$. For this, by Hasse–Minkowski Theorem we have to check that the quadratic form N_{red} on $D_0:=\{x\in D: x+x^\sigma=0\}$ represents d locally. But, this is obvious for the places v where D_v splits because $N_v: D_v\to K_v$ is hyperbolic ([18], 2.11.10). If D_v does not split, we use our assumption that $SU(V_v,h_v)$ is isotropic to write $(V_v,h_v)=(\alpha_v,-\alpha_v,\beta_v)$ and so $d=N(\beta_v)$. So, we have some $\lambda\in D_0^*$ such that $N(\lambda)=d$. We are left with showing the existence of $e\in V$, with $h(e)=\lambda$, i.e. that (V,h) represents λ. Again, by ([18], 10.4.1), we must show such a representation locally. At all places v where D_v is a division algebra, this holds by ([18], Ths. 10.3.6, 10.3.7). At finite places v where D_v is split also, it holds good since $V_\lambda:=V\perp\langle-\lambda\rangle$ becomes Morita equivalent (by Remark 9.4) to a quadratic form in 8 variables over K_v, which therefore has Witt index $\geqslant 2$. At real places v where D_v is split, we choose $\varepsilon_v=\pm 1$ such that $V_v\perp\langle-\varepsilon_v\lambda\rangle$ has Witt index $\geqslant 2$. By weak approximation at these real places, we get $t\in K$ so that V represents $t\lambda$. Thus, $\exists e\in V$ with $N_{\text{red}}h(e)=d$ in K^*/K^{*2}. To continue the proof of Th. 10.8, we shall apply induction on $\dim V$. For $\dim V=3$, we choose as in the lemma above $e\in V$ such that $W=(eD)^\perp$ has the property that W_v is universal in V_v for almost all v. Indeed, if $T\supseteq\{v: SU(W_v,h_v) \text{ is compact}\}$, then W_v is universal in V_v, for v not in T.

Lemma 10.10. *Let* $v\in T, e\in V$. *Put* $W=(eD)^\perp$. *Then,* $\exists g_n\in G(V_v)$ *such that* $h(e-g_ne)\in h_v(W_v)$ *and* $g_n\to 1$ *T-adically.*

Proof. Recall the definition of a quasi-reflection (after Remark 9.4); viz. for $a\in V, \alpha\in D$, we have $\tau_{a,\alpha}(x)=x-a\alpha h(a,x)$. It is in $U(V,h)$ if $h(a)=\alpha^{-1}-(\alpha^{-1})^\sigma$ i.e. if $\alpha^{-1}\in h(a)/2+K$. Consider any $w\in W_v$. Let $w_n\in V_v\setminus W_v$ such that $w_n\to w$. So, $W_v\perp\langle-h(w_n)\rangle$ is isotropic for $n\gg 0$, since $W_v\perp\langle-h(w)\rangle$ is. Therefore $h(w_n)\in h(W_v)$ for $n\gg 0$. Consider $\alpha_n^{-1}=h(w_n)/2+t_n$, where $t_n\in K_v^*$, $t_n\to 0$. Put $g_n=\tau^2_{w_n,\alpha_n}$. Then, $e-g_ne=w_n\beta_n$, say, where we have written β_n for $2-\alpha_nh(w_n)\alpha_nh(w_n,e)$. Hence $h(e-g_ne)\in h(W_v)$ for $n\gg 0$. Note, moreover, that $g_n\to 1$ since $\alpha_n\to 2h(w)^{-1}$ and $h(w_n)\to h(w)$. This proves the lemma.

Continuing the proof of Theorem 10.8, let $a\in N^\dagger$. Using the lemma, we get $\forall v\in T$

Get $f_n \in W = (eD)^{\perp}$ such that $h(f_n) = h(e - gx_n \cdot e)$. Then, $gx_n \cdot e = g_n \cdot e$, where g
$\tau_{e-gx_n\cdot e,\alpha_n}\tau_{f_n,\alpha_n}\cdot e \in G(W_n)$ for some $W_n \subseteq V$ of dimension 2, viz. $W_n = (e - gx_n e)D +$
We note firstly that for $v \in T$, and any $X \subseteq V_v$ of dimension $\geqslant 2$, the orbit
$G(X) \to G(X)\cdot e$ is open. For every $v \in T$, we apply this simultaneously to the o
maps under all the $G(W_{n,v})$, and get $h_n^v \in G(W_{n,v})$ such that $gx_n \cdot e = h_n^v \cdot e$ and h_n^v
Since the torus $G(f_n D)$ splits over a quadratic extension, the weak approximatio
valid for it, and hence we can get $g_n \in G(W_n)$ such that $gx_n \cdot e = g_n \cdot e$ and $g_n \to 1$ *T-adica*
Thus, $g_n^{-1} gx_n \in N$ for $n \gg 0$. Hence $g_n^{-1} g \in N$ for $n \gg 0$. We derive this statement
dim $V \geqslant 4$ as follows. For dim $V \geqslant 4$, we choose by lemma 10.9, $e \in V$ such
$(eD)^{\perp} \supseteq W$ where dim $W = 3$ and $SU(W,h)$ is locally isotropic at all finite place
K. Let $g \in N^{\dagger}$. Letting $T \supseteq \{v: SU(W_v)$ is compact$\}$, we will get $x_n \in N$ such that gx_n
T-adically. Without loss of generality, we can assume that $gx_n \cdot e \neq e$ for otherwise
are done by induction since $gx_n \in G((eD)^{\perp}, h)$ and $\to 1$ *T-adically*. Then, $h(e - gx$
$h_v(eD_v)^{\perp} \forall v$. By the local-global principle once again, $h(e - gx_n e) \in h(eD)^{\perp}$. He
$gx_n \cdot e = g_n \cdot e$ where $g_n = \tau_{e-gx_n e,\alpha_n} \cdot \tau_{f_n,\alpha_n} \in G(W_n)$ for some $W_n = (e - gx_n e)D + f_n D$
dimension 2 where $f_n \in (eD)^{\perp}$ with $h(f_n) = h(e - gx_n e)$. Proceeding as before, we
get g_n as above which also $\to 1$ *T-adically*. Thus, $g_n^{-1} gx_n \in G((eD)^{\perp})$ and $\to 1$ *T-adica*
So, by the induction hypothesis, $g_n^{-1} gx_n \in N$. Hence, once again we get $g_n^{-1} g \in N$
$n \gg 0$. We want to show that $g_n \in N$ for $n \gg 0$. Now, recall that $g_n \cdot e = g_n' \cdot e$
$g_n' = \tau_{e-gx_n\cdot e}\tau_{f_n} \in [SU(e_n D), SU(f_n D)]$ for some e_n, f_n. So, $g_n^{-1} g_n' \in G((eD)^{\perp}) \cap G(W_n$
$G(h_n D)$ for some h_n. To summarise, any element g in $N^{\dagger}$ is, modulo N, of the fo
$g_1 g_2$ with $g_1 \in [SU(xD), SU(yD)], g_2 \in SU(zD)$ where $x, y, z \in X$, $X \subseteq V$ of dimensio
and moreover g is *T-adically* as deep as we want i.e. we can assume that $g \in G(X_v) \cap$
$\forall v \in T$ where

$$N \cap G(X) \supseteq G(X) \cap \prod_T (G(X_v) \cap W_m^v) \cdot \prod_{S_X \setminus T} [G(X_v), G(X_v)].$$

Now, we can assume that $S_X \setminus T \neq \varnothing$, for otherwise we are through. Taking T
contain all v such that D_v is a division algebra, we have $\forall v \in S_X \setminus T$, $G(X_v) \cong A^1 \times$
$\{\pm(1,1)\}$ for the unique quaternion division algebra A over K_v. Now, we can wr
g in $G(X_v)$ as $g = [\alpha_v, \beta_v]\cdot[\gamma_v, \delta_v] \in [SU(xD_v), SU(X_v)]\cdot[SU(t_v D_v), SU(X_v)]$ for so
$t_v \in X_v$. (In fact any element of $G(zD_v)$ can be written as $[(\pi_v, \pi_v), (a_v, b_v)]$). We
approximating at the places in $S_X \setminus T$ for t_v, and the elements $\alpha_v, \beta_v, \gamma_v, \delta_v$) and at
places T for the identity, we can get $g = [\alpha, \beta]\cdot[\gamma, \delta]$ modulo $N \cap G(X)$, wh
$\alpha \in SU(xD)$, $\gamma \in SU(tD)$, $\beta, \delta \in SU(X)$ and $\alpha, \beta, \gamma, \delta$ are *T-adically* deep. By doing
approximation at real places too, we can suppose that the local spinor nor
$\theta_v(\beta) = \theta_v(\delta) = 1$ for all real v. So, we can choose $\eta \in SU((xD)^{\perp}), \phi \in SU((tD)^{\perp})$ w
$\theta(\eta) = \theta(\beta)$, $\theta(\phi) = \theta(\delta)$ by the following proposition:

PROPOSITION 10.11.

Let (W, h) be a skew-Hermitian space over a quaternion algebra over K, of dimensi
$\geqslant 2$. Then, if $t \in K^*$ is such that $t > 0$ at all the real completions, then $\exists h \in SU(W$
with $\theta(h) = t$.

Proof. Consider $1 \to \mu_2 \to \mathrm{Spin}_W \to SU_W \to 1$ where we have written Spin_W etc. for t
algebraic group defined over K by (W, h). We have the corresponding Gal

cohomology sequence

$$1 \to \{1, -1\} \to \mathrm{Spin}(W) \to SU(W) \to H^1(K, \mu_2) \to H^1(K, \mathrm{Spin}_w).$$

Let us denote the last two maps above by θ and $*$ respectively (indeed, θ is the spino norm map). Now, we know by [5] that for a finite place v of K, $H^1(K_v, \mathrm{Spin}_W) = \{1\}$ So, we get from the Hasse principle [5] that $H^1(K, \mathrm{Spin}_W) \to \Pi_v H^1(K_v, \mathrm{Spin}_W)$ i injective, where the product is taken over real places. Thus, $*(t) = 1$ and so $t = \theta($ for some h.

To finish the proof of the theorem for the types ${}^2A_3, {}^1D_n, {}^2D_n$, we note that th spinor norm map on $SU(V_v)$ is an open map and so, we can get η, ϕ such tha $\beta\eta, \delta\phi \in N^\dagger$. So, $g = [\alpha, \beta\eta] \cdot [\gamma, \delta\phi] \in [SU(xD), N^\dagger] \cdot [SU(tD), N^\dagger] \subseteq N$. Therefore $g \in N$

Proof of Th. 10.8 *for type* B_l. We write $G(K) = \mathrm{Spin}(V, h)$ for a quadratic space ove K of dimension $\geqslant 5$ and will apply induction on dim V. Let $g \in N^\dagger$, $e \in V$.

Notice that $S = \varnothing$.

Start with any finite set T of finite places of K containing $\{v$ finite: $SO((eK_v)^\perp)$ i compact$\}$. We can get $x_n \in N$ such that $gx_n \to 1$ *T-adically*. Now, $gx_n \cdot e = \tau_{e_n} \cdot e$ wher $e_n = e - gx_n \cdot e$. Since a 4-dimensional quadratic form over a non-Archimedean loca field is universal, we can get on using Hasse–Minkowski theorem a vector $f_n \in (eK)$ such that $h(e_n) = h(f_n)$. By weak approximating, we get $gx_n \cdot e = h_n \cdot e$ where $h_n \in G(X_n$ $\to 1$ *T-adically* where dim $W_n = 4$. Since T contains all those finite places v such tha $SO((eK_v)^\perp)$ is compact, we get by induction that $h_n^{-1} gx_n \in N$ for $n \gg 0$. Hence, modul N, we have that $g \in G(X)$ and is *T-adically* deep for a 4-dimensional X with

$$N \cap G(X) \supseteq G(X) \cap \prod_T (G(X_v) \cap W_m^v) \cdot \prod_{S_X \setminus T} [G(X_v), G(X_v)].$$

Hereafter, the proof is the same as the case of D_l, as we have $G(X_v) \cong A^1 \times A^1/\{\pm Id$ for $v \in S_X \setminus T$ where A is the quaternion division algebra over K_v. The analogue Proposition 10.11 is still valid.

Remark. In our Proofs for types A_3, B_l, D_l we have been working with the grou $G(V) = G(K)/\{\pm 1\}$ rather than with $G(K)$. So, a priori our proofs show only that N is a non-central normal subgroup of Γ, then $N \cdot \{\pm 1\}$ is S-adically open. But, w can easily deduce that N itself is S-adically open as follows. We need only show tha $N \cap U$ is S-adically open for some S-adic open U. We can choose U so deep at som place in S that -1 is not in U. Then, $(N \cap U) \cdot \{\pm 1\} \supseteq D$ for some S-adic open $D \subseteq U$ Consider the canonical map $\phi: G(K) \to G(K)/\{\pm 1\}$. Let $d \in D$. Then, $\phi(d) = \phi(n)$ fo some $n \in N \cap U$. So, $n^{-1} d \in \{\pm 1\} \cap U = \{1\}$. Thus, $D \subseteq N \cap U$.

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ontinuous maps between Grassmann manifolds

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Abstract. Let $G_{n,k}$ denote the Grassmann manifold of k-planes in $\mathbb{R}^n$. We show that for any continuous map $f\colon G_{n,k} \to G_{n,l}$ the induced map in $Z/2$-cohomology is either zero in positive dimensions or has image in the subring generated by $w_1(\gamma_{n,k})$, provided $1 \leqslant l < k \leqslant [n/2]$ and $n \geqslant k + 2l - 1$. Our main application is to obtain negative results on the existence of equivariant maps between oriented Grassmann manifolds. We also obtain positive results in many cases on the existence of equivariant maps between oriented Grassmann manifolds.

Keywords. Grassmann manifolds; Steenrod squares; Stiefel–Whitney classes; equivariant maps.

in results

$\leqslant k < n$, let $G_{n,k}$ be the Grassmann manifold of all k-dimensional vector aces ("k-planes") of $\mathbb{R}^n$, and let $\tilde{G}_{n,k}$ denote the oriented Grassmann manifold oriented k-planes in $\mathbb{R}^n$. The double covering map $\pi_{n,k}\colon \tilde{G}_{n,k} \to G_{n,k}$ is a universal ng projection for $n \geqslant 3$. Let $\gamma_{n,k}$ denote the canonical k-plane bundle over $G_{n,k}$ t $\beta_{n,k}$ denote the "orthogonal complement" bundle over $G_{n,k}$, which is of rank Let $w_i = w_i(\gamma_{n,k}) \in H^i(G_{n,k};\, Z/2)$, $1 \leqslant i \leqslant k$, denote the i-th Stiefel-Whitney class , and let $\bar{w}_j = w_j(\beta_{n,k})$, $1 \leqslant j \leqslant n-k$. Then, one has the following relation:

$$w \cdot \bar{w} = (1 + w_1 + \ldots + w_k)(1 + \bar{w}_1 + \ldots + \bar{w}_{n-k}) = 1 \tag{1}$$

can be used to express the $\bar{w}_j$'s in terms of the w_i's.

e has the following description for the $Z/2$-cohomology ring $H^*(G_{n,k})$, (cf. [2],

$(G_{n,k})$ is generated as an algebra over $Z/2$ by $w_1, \ldots, w_k$ subject only to the ons coming from (1). (2)

te that $G_{n,k} \cong G_{n,n-k}$ so we assume without loss of generality that $k \leqslant [n/2]$.

em 1. *Let $1 \leqslant l < k \leqslant [n/2]$, $n \geqslant k + 2l - 1$, and let $2^{s-1} < n \leqslant 2^s$. If $f\colon G_{n,k} \to G_{n,l}$ continuous map, then $f^*\colon H^*(G_{n,l}; Z/2) \to H^*(G_{n,k}; Z/2)$ is*

o in positive dimensions except possibly when $n = 2^s$, or $(n,k) = (2^s - 1, 2)$,

(ii) *either zero in positive dimensions or* $f^*(w_i(\gamma_{n,l})) = \begin{bmatrix} l \\ i \end{bmatrix} w_1^i$, $1 \leqslant i \leqslant l$, *if* $n = 2^s$

$n = 2^s - 1$, $k = 2$.

Note that $G_{\infty,k} = \mathrm{BO}(k)$, a classifying space for real vector bundles of rank k. We will prove

Theorem 2. *Let* $1 \leqslant l < k$. *Then for any* f: $\mathrm{BO}(k) \to \mathrm{BO}(l)$, *the induced map* $Z/2$-*cohomology is zero in positive dimensions or is given by*

$$f^*(w_j(\gamma_{\infty,l})) = \begin{bmatrix} r \\ j \end{bmatrix} w_1(\gamma_{\infty,k})^j \text{ if } j \leqslant r \text{ and } f^*(w_j(\gamma_{\infty,l})) = 0$$

if $j > r$, *for some* $r \leqslant l$. *Conversely, given any* $r \leqslant l$, *there exists a map* f_r: $\mathrm{BO}(k) \to \mathrm{B}$
such that

$$f^*(w_j(\gamma_{\infty,l})) = \begin{bmatrix} r \\ j \end{bmatrix} w_1(\gamma_{\infty,k})^j \text{ for } 1 \leqslant j \leqslant r.$$

We will apply Theorem 1 mainly to the question of existence of equivariant m between oriented Grassmann manifolds. We hope that the above theorems will l to other applications also.

The involution that changes the orientation on each element of $\tilde{G}_{n,k}$ is a smo $Z/2$ action on $\tilde{G}_{n,k}$. Observe that the "inclusions" i: $\tilde{G}_{n,k} \to \tilde{G}_{n+1,k}$, and j: $\tilde{G}_{n,}$ $\tilde{G}_{n+1,k+1}$ induced by the usual inclusion of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ are equivariant. From thi follows that if g: $\tilde{G}_{m,k} \to \tilde{G}_{n,l}$ is equivariant with $m > n$, then there exist equivari maps g_p: $\tilde{G}_{n,p} \to \tilde{G}_{n,l}$, for $k - m + n \leqslant p \leqslant k$, as can be seen from composing g with and j's suitably, $(m-n)$ times. The following theorem is a partial answer to question: "For what values of n, k and l does there exist an equivariant map $\tilde{G}_{n,k} \to \tilde{G}$ $1 \leqslant l$, $k < n$, $l \neq k, n-k$"? Since the diffeomorphism $\perp$: $\tilde{G}_{n,k} \to \tilde{G}_{n,n-k}$ is equivariant, need only consider the case when k, $l \leqslant [n/2]$. When $k = l$, the identity map equivariant.

Theorem 3. *Let* $2^{s-1} < n \leqslant 2^s$.

(i) *Let* $1 \leqslant l < k \leqslant [n/2]$, *and let* $n \geqslant k + 2l - 1$. *Then there does not exist an equivari map of* $\tilde{G}_{n,k}$ *into* $\tilde{G}_{n,l}$ *provided that* l *is even when* n *is a power of* 2, *and t* $(n,k) \neq (2^s - 1, 2)$.
(ii) *Let* $m \leqslant 2^{s-1}$, $1 \leqslant p \leqslant [m/2]$, $1 \leqslant k \leqslant [n/2]$. *Then there does not exist an equivari map of* $\tilde{G}_{n,k}$ *into* $\tilde{G}_{m,p}$. *There does not exist an equivariant map of* $\tilde{G}_{n,k}$ *into* $\tilde{G}_{2^s-}$ *for* $k \geqslant 3$, $(n,k) \neq (2^{s-1}+1, 3)$. *No equivariant map exists from* $\tilde{G}_{n,2}$ *or* $\tilde{G}_{2^{s-1}+1,3}$ *i* $\tilde{G}_{2^s-2,1}$.

As for positive results we prove

Theorem 4. (i) *Let* $1 \leqslant k \leqslant [n/2]$, *and let* $d = \dim \tilde{G}_{n,k} = k(n-k)$. *For any* r, *there ex an equivariant map* f: $\tilde{G}_{n,k} \to \tilde{G}_{d+r,q}$, $1 \leqslant q \leqslant r$. *Further, when* $(n,k) \neq (2^s+1, 2)$ *th exists an equivariant map* f': $\tilde{G}_{n,k} \to \tilde{G}_{d-1+r,q}$, $1 \leqslant q \leqslant r$. *There does not exist equivariant map from* $\tilde{G}_{2^s+1,2} \to \tilde{G}_{d,1}$.

$S^{n-1} \cong \tilde{G}_{n,1} \to \tilde{G}_{n,k}$ *for* k *odd and* $1 \leqslant k \leqslant 8a + 2^b$, *where* $n = 2^{4a+b}$. *(odd),* $0 \leqslant b \leqslant 3$, $a \geqslant 0$. *Conversely, if there exists an equivariant map* $S^{n-1} \to \tilde{G}_{m,k}$, $2k \leqslant m \leqslant n$, *then* $m = n$, k *is odd and* $k \leqslant 2^{4a+b}$.

Recall that the span of a smooth manifold M is the maximum number of linearly independent (tangential) vector fields that M admits. It was known [5] since 1985 that $3 \leqslant \text{Span } G_{6,3} \leqslant 7$. As a further application we prove, as a corollary to Theorem 5 proved in § 4 that

Theorem 6. *Span* $G_{6,3} = 7$.

Theorems 1, 2 and 3 are proved in § 2, Theorem 4 in § 3, and Theorems 5 and 6 in § 4. The following result of Stong [13] will be used in the proofs. Recall that for $w_1 \in H^1(G_{n,k}; Z/2)$, $ht(w_1)$: $=$ height $(w_1) = \sup\{m | w_1^m \neq 0\}$.

Theorem 7. (*Stong* [13]). *Let* $2 \leqslant k \leqslant n/2$, $2^{s-1} < n \leqslant 2^s$. *Then*

$$ht(w_1) = \begin{cases} 2^s - 2 \text{ if } k = 2; \text{ or } n = 2^{s-1} + 1,\ k = 3 \\ 2^s - 1 \text{ otherwise.} \end{cases} \qquad \square$$

Unless otherwise mentioned, throughout this paper all cohomology groups will have mod 2 coefficients.

2. Proofs of theorems 1, 2 and 3.

Our proofs are quite elementary as they make use of only basic properties of characteristic classes of vector bundles.

Recall, first, that the Steenrod operations on $w_p(\xi)$, the p-th Stiefel-Whitney class of any vector bundle ξ are given by the Wu relations [10]:

$$Sq^j(w_p(\xi)) = \sum_{0 \leqslant i \leqslant j} \begin{bmatrix} p - j + i - 1 \\ i \end{bmatrix} w_{p+i}(\xi) w_{j-i}(\xi), \tag{3}$$

with the usual conventions that $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1$ and $w_0(\xi) = 1$. In particular, if $w_q(\xi) = 0$ for all $q > r$, then

$$Sq^j(w_r(\xi)) = w_r(\xi) w_j(\xi),\ 1 \leqslant j \leqslant r. \tag{4}$$

Secondly, notice from (2) that there are no algebraic relations among $w_1, \ldots, w_k$ in $H^*(G_{n,k})$ in dimensions $\leqslant n - k$. Assume $2k \leqslant n$, and order the w_i's by declaring that $1 = w_0 < w_1 < \ldots < w_k$. Extend this to a simple ordering of all monomials of total degree $\leqslant n - k$, as follows: Let $I = i_1, \ldots, i_p$; $J = j_1, \ldots, j_p$ be non-increasing sequences of non-negative numbers with $i_1, j_1 \leqslant k$. We write $w_I = w_{i_1} \ldots w_{i_p}$. Declare that $w_I > w_J$ if for some r, $0 < r \leqslant p$, $i_1 = j_1, \ldots, i_{r-1} = j_{r-1}$ and $i_r > j_r$. For example, when $n = 20$, $k = 7$, we have $w_7 w_6 > w_7 > w_6^2 > w_6 w_1^5 > w_4^2 w_3 > w_2^5 w_1 > w_2 w_1^5 > w_1^8$.

For $0 \neq a \in H^*(G_{n,k})$, deg $(a) \leqslant n - k$, we denote the largest monomial that occurs in a with coefficient $1 \in Z/2$ by $L(a)$, and define $L(0) = 0$. It is easy to check that if $b \in H^*(G_{n,k})$, deg $(a) + \deg(b) \leqslant n - k$, then

In particular, if $b = w_I$, then

$$L(aw_I) = L(a)\cdot w_I.$$

We caution the reader that homomorphisms in $H^*(\ \ ;Z/2)$ induced by a cont map between Grassmann manifolds need not be order preserving. Howev ordering turns out to be a useful tool in our proofs.

Proof of Theorem 1. Write $v_i = w_i(\gamma_{n,l})$, and let $u_i = w_i(f^*(\gamma_{n,l})) = f^*(w_i(\gamma_{n,l})) =$ $1 \leqslant i \leqslant l$. Then $u_i = h_i(w_1, \ldots, w_i)$ for a suitable polynomial h_i, where $w_j = w_j(\gamma_n$ r be the largest integer for which $u_r \neq 0$, and assume $r \geqslant 1$. Write u_r as

$$u_r = w_m^p g_0 + w_m^{p-1} g_1 + \cdots + g_p$$

where $m \geqslant 1$ is the largest integer such that w_m occurs in the expression of polynomial in $w_1, \ldots, w_r$, and $g_i = g_i(w_1, \ldots, w_{m-1})$, $0 \leqslant i \leqslant p$, with $g_0 \neq 0$.

At this stage we comment that the proof involves two steps. First we sho $\operatorname{Im} f^*$ is contained in the subring generated by w_1 (see Claim below). Then Stong's result on the height of w_1 (Theorem 7) we show that f^* is as asserted theorem.

Claim: $m = 1$. To get a contradiction, assume that $m > 1$. Then $2m - 1 > m$.

Case 1: Let $2m - 1 \leqslant k$. Then by (3), $L(Sq(w_j)) = L(Sq^{j-1}(w_j)) = w_{2j-1}$, for Note that $(m-1)p + mp \leqslant 2r - 1 \leqslant n - k$. Now consider

$$\begin{aligned} Sq^{(m-1)p}(w_m^p) &= (Sq^{(m-1)}(w_m))^p + \text{ terms involving } w_m^2 \\ &= (w_{2m-1} + \cdots + w_m w_{m-1})^p + \text{ terms involving } w_m^2. \\ &= w_{2m-1}^p + \text{ lower terms.} \end{aligned}$$

Again $(m-1)p + r \leqslant 2r - 1 \leqslant n - k$. Now

$$\begin{aligned} Sq^{(m-1)p}(u_r) &= Sq^{(m-1)p}(w_m^p g_0) + \text{ terms smaller than } w_{2m-1}^p, \\ &= Sq^{(m-1)p}(w_m^p) g_0 + \text{ terms smaller than } w_{2m-1}^p, \\ &= w_{2m-1}^p g_0 + \text{ terms smaller than } w_{2m-1}^p. \end{aligned}$$

Therefore if $L(g_0) = w_I$ and $L(u_{(m-1)p}) = w_J$, then by comparing both sides equality $L(Sq^{(m-1)p}(u_r)) = L(u_r u_{(m-1)p})$, we get

$$w_{2m-1}^p w_I = L(Sq^{(m-1)p}(u_r)) = L(u_r u_{(m-1)p}) = w_m^p w_I w_J.$$

This is a contradiction as $2m - 1 > m$ and $(m-1)p + r \leqslant n - k$, and there a algebraic relations among the w_i's in dimensions up to $n - k$. This shows that in case $2m - 1 \leqslant k$.

where $g_0 \neq 0$, f_i, $0 \leqslant i \leqslant m-j$, are polynomials in $w_1, \ldots, w_{j-1}$ only by dimensio considerations. (Some of the f_i's can be zero.)
For $j \leqslant \alpha < m$,

$$\begin{aligned} Sq^{r-1}(w_\alpha f_{m-\alpha}) &= Sq^{\alpha-1}(w_\alpha) f_{m-\alpha}^2 + w_\alpha^2 Sq^{r-\alpha-1}(f_{m-\alpha}) \\ &= (w_k w_{2\alpha-1-k} + \cdots + w_\alpha w_{\alpha-1}) f_{m-\alpha}^2 + w_\alpha^2 Sq^{r-\alpha-1}(f_{m-\alpha}) \\ &= w_k w_{2\alpha-1-k} f_{m-\alpha}^2 + \text{ terms not involving } w_k. \end{aligned}$$

Therefore,

$$Sq^{r-1}(u_r) = w_k(w_{2m-1-k} g_0^2 + \Sigma w_{2\alpha-1-k} f_{m-\alpha}^2) + \text{ terms not involving } w_k$$

The coefficient of w_k in the above is non-zero because in $w_{2m-1-k} g_0^2$, w_{2m-1} occurs with odd exponent whereas in other terms it occurs, if at all, with even expone Therefore w_k divides $L(Sq^{r-1}(u_r))$. As in the previous case this leads to a contradictio

This establishes our claim that $m=1$.

Now if $u_r = w_1^r$, then for $j<r$, $u_j u_r = Sq^j(u_r) = \begin{bmatrix} r \\ j \end{bmatrix} w_1^{j+r} = \begin{bmatrix} r \\ j \end{bmatrix} w_1^j u_r$. As $j<$ $j+r \leqslant 2r-1 \leqslant n-k$, we deduce that $u_j = \begin{bmatrix} r \\ j \end{bmatrix} w_1^j$.

It follows that for the dual Stiefel-Whitney class, $\bar{v}_j$, which is a certain polynom in $v_1, \ldots, v_l$, we must have

$$f^*(\bar{v}_j) = a_j w_1^j, \text{ for some } a_j \in Z/2.$$

Applying f^* to the relation (1) for the bundle $\gamma_{n,l}$, we see that

$$\left(1 + \begin{bmatrix} r \\ 1 \end{bmatrix} w_1 + \cdots + \begin{bmatrix} r \\ r \end{bmatrix} w_1^r \right)(1 + a_1 w_1 + \cdots + a_{n-l} w_1^{n-l}) = 1.$$

If p is the largest integer for which $a_p = 1$, then we get $w_1^{r+p} = 0$. But $r \leqslant l, p \leqslant n-l$ $r+p \leqslant n$. Thus $w_1^n = 0$. This contradicts Theorem 7, the result of Stong [13] on height of w_1 unless $n = 2^s$ and $r = l$; or $n = 2^s - 1$ and $k = 2$. This implies that in c $n \neq 2^s$, and $(n,k) \neq (2^s - 1, 2)$, we must have $r = 0$. It follows that $u_i = 0$ for all $i \geqslant$ Since $H^*(G_{n,l})$ is generated by $v_1, \ldots, v_l$, we see that f^* is zero in positive dimensio

If $n = 2^s$, and $r \geqslant 1$, then $r = l$. Thus $u_l = w_1^l$, and $u_i = \begin{bmatrix} l \\ i \end{bmatrix} w_1^i$. If $n = 2^s - 1$, $k = 2$, th $l = 1$, and, $f^*(v_1) = w_1$, when f^* is non-zero in positive dimensions.

Proof of Theorem 2. The proof of our claim that $m = 1$ in the above holds eve $n = \infty$. Hence, writing $v_i = w_i(\gamma_{\infty,l})$ and $w_i = w_i(\gamma_{\infty,k})$, for any f: BO$(k) \to$ BO

that is $f^*(\gamma_{\infty,l})=\eta$. Then $f^*(w(\gamma_{\infty,l}))=w(\eta)=(w(\zeta))^r=(1+w_1)^r$. Hence, $f^*(v_i)=\binom{r}{i}w_1^i$, $1\leqslant i\leqslant r$, and $f^*(v_j)=0$ for $j>r$. □

NOTE: R R Patterson [11] has characterized all algebra homomorphisms from $H^*(\mathrm{BO};Z/2)$ to $H^*(\mathrm{BO}(k);Z/2)$ which respect the Steenrod operations.

Proof of Theorem 3. (i) It is easy to see that if g: $\tilde{G}_{n,k}\to\tilde{G}_{n,l}$ is equivariant then for the induced map f: $G_{n,k}\to G_{n,l}$, $f^*(w_1(\gamma_{n,l}))=w_1(\gamma_{n,k})$. Therefore it follows from Theorem 1, that either $n=2^s$ and l is odd, or $n=2^s-1$, and $k=2$, $l=1$.
(ii) Assume the contrary and consider the induced map between the Grassmann manifolds. A contradiction is obtained on comparing the heights of w_1 using Theorem 7 stated in the introduction. □

3. Construction of equivariant maps

Let $\zeta_{n,k}$ (or simply ζ) denote the unique (up to bundle isomorphism) non-trivial line bundle over $G_{n,k}$, so that $w_1(\zeta_{n,k})=w_1\in H^1(G_{n,k})\cong Z/2$. Notice that a map f: $G_{n,k}\to G_{m,p}$ is covered by an equivariant map $\tilde{f}$: $\tilde{G}_{n,k}\to\tilde{G}_{m,p}$ (i.e. $\pi_{m,p}\circ\tilde{f}=f\circ\pi_{n,k}$) if and only if $f^*(\zeta_{m,p})\approx\zeta_{n,k}$, or equivalently $f^*(w_1(\gamma_{m,p}))=w_1(\gamma_{n,k})$. Another useful characterization of the existence of equivariant maps between oriented Grassmannians in terms of vector bundles over Grassmann manifolds is the following:

PROPOSITION 3.1.

There exists an equivariant map $\tilde{f}$: $\tilde{G}_{n,k}\to\tilde{G}_{m,p}$ if and only if there exist vector bundles ξ and η over $G_{n,k}$ of ranks p and $(m-p)$ respectively such that ξ is non-orientable and $\xi\oplus\eta\approx m\varepsilon$.

Proof. Suppose that $\tilde{f}$:$\tilde{G}_{n,k}\to\tilde{G}_{m,p}$ is equivariant and f: $G_{n,k}\to G_{m,p}$ is the map induced by $\tilde{f}$. Let $\xi=f^*(\gamma_{m,p})$, and $\eta=f^*(\beta_{m,p})$. Then $\xi\oplus\eta\approx f^*(\gamma_{m,p}\oplus\beta_{m,p})\approx f^*(m\varepsilon)\approx m\varepsilon$. Also since $\tilde{f}$ is equivariant, $w_1(\xi)=f^*(w_1(\gamma_{m,p}))=w_1(\gamma_{n,k})\neq 0$. Hence ξ is non-orientable.

Conversely, assume that ξ^p and η^{m-p} are vector bundles over $G_{n,k}$ with ξ non-orientable and $\xi\oplus\eta\approx m\varepsilon$. Choose a trivialization φ: $E(\xi\oplus\eta)\to G_{n,k}\times\mathbb{R}^m$. Define f: $G_{n,k}\to G_{m,p}$ by $f(V)=pr_2\circ\varphi(F_\xi(V))$, where $F_\xi(V)$ denotes the fibre of ξ over $V\in G_{n,k}$, and pr_2 is the projection $G_{n,k}\times\mathbb{R}^m\to\mathbb{R}^m$. Then f is continuous, and $f^*(\gamma_{m,p})=\xi$. Since ξ is non-orientable, it follows that $w_1(\gamma_{n,k})=w_1(\xi)=w_1(f^*(\gamma_{m,p}))=f^*(w_1(\gamma_{m,p}))$. Hence f is covered by an equivariant map $\tilde{f}$: $\tilde{G}_{n,k}\to\tilde{G}_{m,p}$.

Proof of Theorem 4(i). Recall from Theorem 1.2 Ch. 8 [4], that any real vector bundle

Tensoring both sides by ζ and observing that $\zeta \otimes \zeta \approx \varepsilon$ as ζ is a line bundle, we obtain

$$(d+1)\varepsilon \approx (\eta \otimes \zeta) \oplus \zeta.$$

Applying Proposition 3.1 we see that there exists an equivariant map

$$h\colon \tilde{G}_{n,k} \to \tilde{G}_{d+1,1}.$$

For any r, and q, $1 \leqslant q \leqslant r$, one obtains an equivariant map $\tilde{G}_{n,k} \to \tilde{G}_{d+r,q}$ by suitably composing with h the equivariant inclusions i's and j's mentioned in the introduction.

When $(n,k) = (2^s+1, 2)$, from Stong's Theorem, $ht(w_1) = 2^{s+1} - 2 = d$. Therefore there is no map $g\colon G_{n,k} \to G_{d,1} = \mathbb{R}P^{d-1}$ with the property that $g^*(w_1(\gamma_{d,1})) = w_1(\gamma_{n,k})$. Consequently there exists no equivariant map $\tilde{G}_{n,k} \to \tilde{G}_{d,1}$ in this case.

When $(n,k) \neq (2^s+1, 2)$, $2 \leqslant k \leqslant [n/2]$, we see that $w_d(d\zeta) = w_1(\zeta))^d = w_1^d = 0$. The manifold $G_{n,k}$ is orientable if and only if n is even, whereas $d\zeta$ is orientable if and only if d is even. It follows that $w_1(G_{n,k}) \neq w_1(d\zeta)$ if and only if n or k is odd, as $d = k(n-k)$. Applying Proposition 3.10(i) of [6], we see that $d\zeta$ admits a nowhere vanishing section, providing $(n,k) \neq (2^s+1, 2)$ and n or k is odd.

If n and k are both even, then $d = k(n-k)$ is even, and $G_{n,k}$ is orientable. Write $d = 2m$. Then $d\zeta = 2m\zeta = m$-fold Whitney sum of the oriented 2-plane bundle 2ζ. The Euler class $e(2\zeta) \in H^2(G_{n,k}; Z)$ can be shown to be a torsion element. In fact $2e(2\zeta) = 0$ (See Prob. 9A, [9].) It follows that $2e(2m\zeta) = 2\cdot(e(2\zeta))^m = 0$ in $H^{2m}(G_{n,k}; Z) = H^d(G_{n,k}; Z) \cong Z$. Hence $e(2m\zeta) = 0$. Therefore $2m\zeta$ must admit a nowhere vanishing section over the d-skeleton of $G_{n,k}$, which is the whole of $G_{n,k}$. As before we conclude that there exists an equivariant map $\tilde{G}_{n,k} \to \tilde{G}_{d-1+r,q}$, $1 \leqslant q \leqslant r$ in this case, completing the proof of 4(i).

Proof of 4(ii). It can be shown that $\tilde{G}_{4,2} \approx S^2 \times S^2$ (cf. p. 104, [3]) under a $Z/2$-equivariant diffeomorphism. Here the $Z/2$-action on $S^2 \times S^2$ is given by the map $(a,b) \mapsto (-a, -b)$ for $(a,b) \in S^2 \times S^2$. Then the composite

$$\tilde{G}_{4,2} \xrightarrow{\cong} S^2 \times S^2 \xrightarrow{\text{proj}} S^2$$

is $Z/2$-equivariant in our sense.

Using the properties of the 2-fold vector product $v\cdot \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ and the 3-fold vector product $\mu\colon \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$, as given by Zvengrowski [14], we construct equivariant maps $f\colon \tilde{G}_{7,2} \to \tilde{G}_{7,1}$ and $g\colon \tilde{G}_{8,3} \to \tilde{G}_{8,1}$ as follows: if (a,b) is an ordered basis in the orientation on $V \in \tilde{G}_{7,2}$, then $v(a,b) \in S^6 = \tilde{G}_{7,1}$ depends only on the oriented vector space V and not on the specific choice of the basis in the orientation of V. We let $f(V) = v(a,b)$. Then f is continuous. If $-V$ is the same vector space as V but with opposite orientation, then $f(-V) = v(-a,b) = -v(a,b) = -f(V)$ (cf. [14]). Hence f is equivariant. The map $g\colon \tilde{G}_{8,3} \to \tilde{G}_{8,1}$ is defined similarly. See [14].

We now construct equivariant maps $S^{n-1} = \tilde{G}_{n,1} \to \tilde{G}_{n,k}$ for k odd and $k \leqslant \rho(n)$ where $\rho(n) - 1$ is the span of S^{n-1}. From Adams [1], $\rho(n) = 8a + 2^b$ where $n = 2^{4a+b}$ (odd), $0 \leqslant b \leqslant 3$, $a \geqslant 0$. If $V_{n,k}$ denotes the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$, the bundle projection $p\colon V_{n,k} \to S^{n-1}$, $p(v_1, \ldots, v_k) = v_1$ admits a $Z/2$-equivariant section $s\colon S^{n-1} \to V_{n,k}$ if $k \leqslant \rho(n)$. Here the $Z/2$ action on $V_{n,k}$ is given by $(a_1, \ldots, a_k) \mapsto (-a_1, \ldots, -a_k)$ for each $(a_1, \ldots, a_k) \in V_{n,k}$. The projection map $q\colon V_{n,k} \to \tilde{G}_{n,k}$ is equivariant if k is odd. Therefore the composite $q \circ s\colon \tilde{G}_{n,1} = S^{n-1} \to \tilde{G}_{n,k}$ is equivariant

Now suppose that h: $G_{n,1} = \mathbb{R}P^{n-1} \to G_{m,k}$ is induced by an equivariant map $S^{n-1} \to \tilde{G}_{m,k}$ with $2k \leqslant m \leqslant n$. Then the map h^*: $H^*(G_{m,k}) \to H^*(\mathbb{R}P^{n-1}) = (Z/2)[x]$ $\langle x^n \rangle$ has the property that $h^*(w_1) = x$, where $w_i = w_i(\gamma_{m,k})$. Let $r \leqslant k$, and $s \leqslant m -$ be the largest integers such that $h^*(w_r) \neq 0$ and $h^*(\bar{w}_s) \neq 0$, respectively. Applyin h^* to the relation $w \cdot \bar{w} = 1$, and comparing $(r+s)$-th degree terms on both sides w obtain $0 = h^*(w_r) \cdot h^*(\bar{w}_s) = x^r \cdot x^s = x^{r+s}$. Hence $r + s \geqslant n$. Since $r + s \leqslant m \leqslant n$, it follow that $r = k$, $s = m - k$ and $m = n$. Now $h^*(w_k) = x^k$ implies, by Wu's formula, tha $h^*(w) = (1+x)^k$. Since $x = h^*(w_1) = \binom{k}{1} x$ we see that k must be odd. Also, $h^*(w) =$ $(1+x)^k$ implies $h^*(\bar{w}) = (1+x)^{-k} = (1+x)^{2^N - k}$ for large enough N, as $(1+x)^{2^N} = 1$ fo N large. This implies that

$$h^*(\bar{w}_j) = \begin{bmatrix} 2^N - k \\ j \end{bmatrix} x^j, \ 1 \leqslant j \leqslant n-k.$$

On the other hand,

$$\begin{aligned} h^*(\bar{w}_j) x^{n-k} &= h^*(\bar{w}_j \bar{w}_{n-k}) = h^*(Sq^j(\bar{w}_{n-k})) \\ &= Sq^j(h^*(\bar{w}_{n-k})) = Sq^j(x^{n-k}) = \begin{bmatrix} n-k \\ j \end{bmatrix} x^{n-k+j}. \end{aligned}$$

Therefore we must have

$$\begin{bmatrix} 2^N - k \\ j \end{bmatrix} \equiv \begin{bmatrix} n-k \\ j \end{bmatrix} \bmod 2, \quad 1 \leqslant j < k.$$

Using Lucas' Theorem [12], p. 5, it can now be seen that if $2^{p-1} < k \leqslant 2^p$, the $n \equiv 0 \bmod 2^p$. This completes the proof. □

Remarks. i) It is possible that Theorems 1 and 3 are true even without the restrictio that $n \geqslant k + 2l - 1$. For $l = 2$, (and $l < k \leqslant [n/2]$) this condition is automaticall satisfied. For $l = 3$, the only exception is the case $k = 4$, $n = 8$ in Theorem 1. In th case one directly shows that Theorem 1 holds.
ii) The question of existence of equivariant maps of $\tilde{G}_{n,k}$ into $\tilde{G}_{n,l}$ in the cas $1 \leqslant k < l \leqslant [n/2]$ seems to be much more difficult to handle, and perhaps requires quite different approach in order to obtain better results than Theorem 4(ii).
iii) Theorem 2 is perhaps well-known to experts in the field, but has been include here for the sake of completeness.

4. Span of $m\zeta_{n,k}$

For a vector bundle ξ and let Span (ξ) denote the maximum number of everywhe

Theorem 5. *Let $2^{s-1} < n \leqslant 2^s$, $2 \leqslant k \leqslant [n/2]$, and let $d = \dim G_{n,k}$.*

i) *Span* $((2^s-2)\zeta_{n,2}) =$ *Span* $((2^s-2)\zeta_{2^{s-1}+1,3}) = 0$, *and for* $k \geqslant 3$, $(n,k) \neq (2^{s-1}+1, 3)$, *Span* $((2^s-1)\zeta_{n,k}) = 0$.
ii) *Span* $(m\zeta_{n,k}) \geqslant$ *Span* $(m\zeta_{d,1})$ *for all* $m \geqslant 1$ *provided* $(n,k) \neq ((2^{s-1}+1), 2)$. *Also, Span* $(m\zeta_{2^{s-1}+1,2}) \geqslant$ *Span* $(m\zeta_{d+1,1})$, and $(d\zeta_{2^{s-1}+1,2}) = 0$.

$$\text{iii) } Span\ (m\zeta_{4,2}) = \begin{cases} 4[m/4] & \text{if } m \equiv 0,1,2 \bmod 4 \\ 4[m/4]+1 & \text{if } m \equiv 3 \bmod 4 \end{cases}.$$

$$\begin{aligned} Span\ (m\zeta_{5,2}) &= Span\ (m\zeta_{6,2}) = Span\ (m\zeta_{7,2}) \\ &= Span\ (m\zeta_{7,1}) = \begin{cases} 8[m/8]+1 & \text{if } m \equiv 7 \bmod 8 \\ 8[m/8] & \text{otherwise} \end{cases}. \end{aligned}$$

$$\begin{aligned} Span\ (m\zeta_{6,3}) &= Span\ (m\zeta_{7,3}) = Span\ (m\zeta_{8,3}) \\ &= Span\ (m\zeta_{8,1}) = 8[m/8]. \end{aligned}$$

Proof: i) $w_{2^s-2}((2^s-2)\zeta_{n,2}) = w_1^{2^s-2} \neq 0$ by Theorem 7. Hence span $((2^s-2)\zeta_{n,2}) = 0$. Other cases follow by the same argument.
ii) Let $(n,k) \neq (2^{s-1}+1, 2)$. As shown in the proof of 4. (i), there exists a map $g = G_{n,k} \to G_{d,1}$ such that $g^*(\zeta_{d,1}) \approx \zeta_{n,k}$. Hence Span $(m\zeta_{n,k}) \geqslant$ Span $(m\zeta_{d,1})$. Similarly Span $(m\zeta_{2^{s-1}+1,2}) \geqslant$ Span $(m\zeta_{d+1,1})$. To show that Span $(d\zeta_{2^{s-1}+1,2}) = 0$ we observe that $d = 2(2^{s-1}-1) = 2^s - 2 = ht(w_1)$ and so $w_d(\zeta_{2^{s-1}+1,2}) = w_1^d \neq 0$.
iii) By 4. (ii), we observe that the following compositions are equivariant.

$$\tilde{G}_{5,2} \overset{i}{\hookrightarrow} \tilde{G}_{6,2} \overset{i}{\hookrightarrow} \tilde{G}_{7,2} \xrightarrow{f} \tilde{G}_{7,1},$$
$$\tilde{G}_{6,3} \overset{i}{\hookrightarrow} \tilde{G}_{7,3} \overset{i}{\hookrightarrow} \tilde{G}_{8,3} \xrightarrow{g} \tilde{G}_{8,1}.$$

Passing to Grassmann manifolds, and pulling back the line bundle $\zeta_{7,1} \approx \gamma_{7,1}$ over $G_{7,1} = \mathbb{R}P^6$ one obtains $\mathrm{Span}(m\zeta_{5,2}) \geqslant \mathrm{Span}(m\zeta_{7,2}) \geqslant \mathrm{Span}(m\zeta_{7,1})$.

From Theorem 1.1 of [7], we obtain $\mathrm{Span}(m\zeta_{7,1})$ to be as stated.

To show that $\mathrm{Span}(m\zeta_{5,2}) = \mathrm{Span}(m\zeta_{7,1})$, we use a Stiefel-Whitney class argument. On $G_{5,2}\, w_1^6 \neq 0$. Therefore for $1 \leqslant m \leqslant 6$, $w_m(m\zeta_{5,2}) = w_1^m \neq 0$.

The proofs for other cases are similar. □

Remark. The above result enables us to determine the order of $[\zeta_{n,k}] \in \mathrm{KO}(G_{n,k})$ for $n \leqslant 8$ except for the case $n = 8$, $k = 4$. For example, $0([\zeta_{7,4}]) = 0([\zeta_{7,3}]) = 0([\zeta_{8,1}]) = 8$.

Proof of Theorem 6. It is well known [8] that a stable normal bundle for the Grassmannian $G_{n,k}$ is $\lambda^2(\gamma_{n,k}) \oplus \lambda^2(\beta_{n,k})$. On $G_{6,3}$, $\gamma = \gamma_{6,3}$ and $\beta = \beta_{6,3}$ are non-orientable 3-plane bundles.

Hence by 10.3, Ch. 12 of [4], $\lambda^2(\gamma) \approx \gamma \otimes \zeta$, $\lambda^2(\beta) \approx \beta \otimes \zeta$ where $\lambda^3(\gamma) = \zeta$ is the non-trivial line bundle over $G_{6,3}$. Hence a stable normal bundle to $G_{6,3}$ is

$$(\gamma \otimes \zeta) \oplus (\beta \otimes \zeta) \approx (\gamma \oplus \beta) \otimes \zeta \approx 6\varepsilon \otimes \zeta \approx 6\zeta.$$

Since by Theorem 5(iii) 8ζ is trivial it follows that the tangent bundle τ of $G_{6,3}$ is stably equivalent to $2\zeta \oplus 7\varepsilon$. Thus stable span of $G_{6,3}$ is 7. It is known due to Korbaš

[5] that Span $G_{6,3} \geqslant 3$. But from Prop. 20.8 and Corollary 20.5 of [6], one has Span $G_{6,3} = 1$ or Span $G_{6,3} =$ stable span of $G_{6,3} = 7$. It follows that Span $G_{6,3} = 7$.

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ntegrability of power series

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Abstract. This paper deals with the integrability of a power series. Our results generalize certain results of Ram, and Askey and Karlin.

Keywords. Integrability of power series; convex function; Jensen's inequality.

roduction

$(x) = \Sigma_0^\infty a_n x^n$ in $[0, 1)$ and let $S_n = \Sigma_{k=0}^n a_k$. In what follows we assume that $\Phi(x)$ ositive increasing and convex function defined on $[0, \infty)$.

1970 Askey and Karlin [1] proved, among others, the following results.

rem A.

$$\int_0^1 \Phi(|f(x)|)\,\mathrm{d}x \leqslant \sum_{n=0}^\infty [(n+1)(n+2)]^{-1}\Phi(|S_n|), \tag{1}$$

$$\int_0^1 (1-x)^\beta \Phi(|f(x)|)\,\mathrm{d}x \leqslant \sum_{n=0}^\infty \Phi(|S_n|)\frac{\Gamma(\beta+2)\Gamma(n+1)}{\Gamma(n+\beta+3)}$$

$$\leqslant K_\beta \sum_0^\infty \Phi(|S_n|)(n+1)^{-\beta-2} \tag{2}$$

> -2.

ese results were subsequently generalized by Ram [2] in the following form.

rem B. *Suppose ψ is a non-negative, non-decreasing function and integrable $L(0, 1)$.*

$$\int_0^1 \psi(x)\Phi(|f(x)|)\,\mathrm{d}x \leqslant K \sum_{n=0}^\infty \Phi(|S_n|)\alpha_n, \tag{3}$$

$\alpha_n = \int_{1-1/n}^1 (1-x)\psi(x)\,\mathrm{d}x$.

r $\psi(x) \equiv 1$, we get Theorem A (1) while for $\psi(x) = (1-x)^\beta$, $0 \geqslant \beta > -2$ we get he case $\beta > 0$ is not included in Theorem B. In order to include this case he also d another theorem.

Theorem C. *Suppose there is an integer $p \geqslant 1$ such that $\psi, \psi', \ldots \psi^{(p-1)}$ are absolutely continuous in $[0, 1)$ and that*

$$\psi(1) = \psi'(1) = \cdots = \psi^{(p-1)}(1) = 0.$$

Furthermore, suppose that $\psi^{(p)}$ has a constant sign and $|\psi^{(p)}|$ is non-decreasing in the set $\{x \in (0, 1)/|\psi^{(p)}(x)| \text{ exists}\}$. Then

$$\int_0^1 \psi(x)\Phi(|f(x)|)\,\mathrm{d}x \leqslant K \sum_{n=0}^{\infty} \Phi(|S_n|)n^{-p-1}\left|\psi^{(p-1)}\left(1 - \frac{1}{n}\right)\right|. \tag{4}$$

Writing $\psi(x) = (1-x)^\beta$, $\beta > 0$ and $p = -[-\beta]$ we get the remaining case of (2).

The aim of this note is to show that it is possible to generalize Theorem B in such a manner that it alone includes both (1) and (2) of Theorem A. Our theorem is as follows:

Theorem 1. *Let $\psi(x)$ be a non-negative function such that $(1-x)\psi(x) \in L(0, 1)$ and $(1-x)^{-\delta}\psi(x)$ is non-decreasing for some $\delta > 0$ in $(0, 1)$. Then*

$$\int_0^1 \psi(x)\Phi(|f(x)|)\,\mathrm{d}x \leqslant K \sum_{n=0}^{\infty} \Phi(|S_n|)\alpha_n, \tag{5}$$

where $\alpha_n = \int_{1-1/n}^1 (1-x)\psi(x)\,\mathrm{d}x$.

It is clear that if $\psi(x)$ is non-decreasing, then $(1-x)^{-\delta}\psi(x)$ is also non-decreasing but the converse need not be true. Thus Theorem B is a corollary of Theorem 1. With $\psi(x) \equiv 1$, Theorem 1 includes (1) and with $\psi(x) = (1-x)^\beta$, $-2 < \beta \leqslant 0$; $\psi(x) = (1-x)^\beta$, $\beta > 0$ (choosing $\delta > \beta$) we deduce (2).

2. For the proof of Theorem 1 we need the following lemma.

Lemma. If $(1-x)\psi(x) \in L(0, 1)$, $\psi(x)$ is non-negative and $(1-x)^{-\delta}\psi(x)$ is non-decreasing for some $\delta > 0$ in $(0, 1)$ then

$$\int_0^1 x^n(1-x)\psi(x)\,\mathrm{d}x \asymp \int_{1-1/n}^1 (1-x)\psi(x)\,\mathrm{d}x.$$

Proof of the lemma.

$$\int_0^1 x^n(1-x)\psi(x)\,\mathrm{d}x = \left(\int_0^{1-1/n} + \int_{1-1/n}^1\right) x^n(1-x)\psi(x)\,\mathrm{d}x$$
$$= L_1 + L_2, \quad \text{say,}$$

we have

$$L_2 \leqslant \int_{1-1/n}^1 (1-x)\psi(x)\,\mathrm{d}x$$

and

$$L_1 = \int_0^{1-1/n} x^n(1-x)^{1+\delta}(1-x)^{-\delta}\psi(x)\,\mathrm{d}x$$

$$\leqslant n^{\delta}\psi(1-1/n)\int_0^{1-1/n} x^n(1-x)^{1+\delta}\mathrm{d}x$$
$$\leqslant Kn^{\delta}\psi(1-1/n)n^{-2-\delta}$$
$$=Kn^{-2}\psi(1-1/n).$$

Since

$$\int_{1-1/n}^{1}(1-x)\psi(x)\mathrm{d}x \geqslant n^{\delta}\psi(1-1/n)\int_{1-1/n}^{1}(1-x)^{1+\delta}\mathrm{d}x=\frac{n^{-2}\psi(1-1/n)}{2+\delta}$$

it follows that

$$L_1 \leqslant K\int_{1-1/n}^{1}(1-x)\psi(x)\mathrm{d}x.$$

Thus

$$\int_0^1 x^n(1-x)\psi(x)\mathrm{d}x \leqslant K\int_{1-1/n}^{1}(1-x)\psi(x)\mathrm{d}x.$$

For the converse part

$$\int_0^1 x^n(1-x)\psi(x)\mathrm{d}x \geqslant (1-1/n)^n\int_{1-1/n}^{1}(1-x)\psi(x)\mathrm{d}x$$
$$\sim e^{-1}\int_{1-1/n}^{1}(1-x)\psi(x)\mathrm{d}x.$$

This proves our lemma.

3. *Proof of Theorem* 1. By Abel's partial summation

$$f(x)=\sum_0^{\infty} S_n x^n(1-x).$$

Using Jensen's inequality, in view of the fact that

$$\sum_0^{\infty} x^n(1-x)=1,$$

we have

$$\Phi(|f(x)|)\leqslant \sum_{n=0}^{\infty}\Phi(|S_n|)x^n(1-x)$$

and consequently by virtue of our lemma

$$\int_0^1 \psi(x)\Phi(|f(x)|)\mathrm{d}x \leqslant \sum_{n=0}^{\infty}\Phi(|S_n|)\int_0^1 x^n(1-x)\psi(x)\mathrm{d}x$$
$$\leqslant K\sum_{n=0}^{\infty}\Phi(|S_n|)\int_{1-1/n}^{1}(1-x)\psi(x)\mathrm{d}x$$
$$=K\sum \Phi(|S_n|)\alpha_n.$$

4. In view of the identities

$$\sum_0^\infty A_n^\alpha x^n = (1-x)^{-\alpha-1},$$

$$f(x) = \sum_0^\infty a_n x^n = (1-x)^{\alpha+1} \sum_0^\infty S_n^\alpha x^n$$

$$S_n^\alpha = \sum_{k=0}^n A_{n-k}^{\alpha-1} S_k = \sum_{k=0}^n A_{n-k}^\alpha a_k$$

$$\sigma_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha}, \quad \alpha > -1,$$

we have

$$|f(x)| \leqslant \sum_0^\infty |S_n^\alpha| x^n (1-x)^{\alpha+1} = \sum_0^\infty |\sigma_n^\alpha| A_n^\alpha x^n (1-x)^{\alpha+1}.$$

Applying Jensen's inequality

$$\Phi(|f(x)|) \leqslant \sum_0^\infty \Phi(|\sigma_n^\alpha|) A_n^\alpha x^n (1-x)^{\alpha+1} \tag{6}$$

and hence

$$\int_0^1 \frac{\psi(x)\Phi(|f(x)|)\,\mathrm{d}x}{(1-x)^{\alpha+1}} \leqslant \sum_{n=0}^\infty \Phi(|\sigma_n^\alpha|) A_n^\alpha \int_0^1 x^n \psi(x)\,\mathrm{d}x$$

$$\leqslant K \sum_{n=0}^\infty \Phi(|\sigma_n^\alpha|) A_n^\alpha \beta_n,$$

where

$$\beta_n = \int_{1-1/n}^1 \psi(x)\,\mathrm{d}x, \quad \psi(x) \in L(0,1).$$

Thus we have established the following result.

Theorem 2. *Suppose ψ is non-negative and integrable $L(0,1)$. If $(1-x)^{-\delta}\psi(x)$ is non-decreasing for some $\delta > 0$, then*

$$\int_0^1 \frac{\psi(x)\Phi(|f(x)|)\,\mathrm{d}x}{(1-x)^{\alpha+1}} \leqslant K \sum_{n=0}^\infty \Phi(|\sigma_n^\alpha|) A_n^\alpha \beta_n,$$

where

$$\beta_n = \int_{1-1/n}^1 \psi(x)\,\mathrm{d}x, \quad \alpha > -1.$$

Choosing $\alpha = 1$ and $\psi(x) = 1$ we get the following result due to Askey and Karlin ([1], Theorem 3)

$$\int_0^1 \Phi(|f(x)|)\,\mathrm{d}x \leqslant K \sum_{n=0}^{\infty} \Phi(|\sigma_n^1|) \tag{7}$$

where $\gamma_n = \int_{1-1/n}^{1} (1-x)^{\alpha+1}\psi(x)\mathrm{d}x$, provided $(1-x)^{\alpha+1}\psi(x) \in L(0,1)$ for $\alpha > -1$ and $(1-x)^{-\delta}\psi(x)$ is non-decreasing for some $\delta > 0$. In particular if $\psi(x) = (1-x)^{-2}$, $\alpha > 0$ we get

$$\int_0^1 \frac{\Phi(|f(x)|)\mathrm{d}x}{(1-x)^2} \leqslant K \sum_{n=0}^{\infty} \Phi(|\sigma_n^\alpha|), \quad \alpha > 0 \tag{9}$$

This gives another estimate for the left side expression in (7).

Writing $\psi(x) \equiv 1$ in (8) we have

$$\int_0^1 \Phi(|f(x)|)\mathrm{d}x \leqslant \sum_{n=0}^{\infty} \Phi(|\sigma_n^\alpha|) A_n^\alpha n^{-\alpha-2}$$

$$\leqslant K \sum_{n=0}^{\infty} \Phi(|\sigma_n^\alpha|) n^{-2}.$$

Taking $\Phi(t) = t^p$, $1 \leqslant p < \infty$ we get

$$\int_0^1 |f(x)|^p \mathrm{d}x \leqslant K \sum_0^\infty \frac{|S_n^\alpha|^p}{(A_n^\alpha)^p} n^{-2}$$

$$\leqslant K \sum_0^\infty |S_n^\alpha|^p n^{-2-\alpha p}, \quad \alpha > -1$$

which is Theorem 4 in [1], where they assume that α is any non-negative integer.

Let

$$f(x) = \sum_0^\infty (-1)^n (n+1) x^n = \frac{1}{(1+x)^2},$$

then $|S_n| \sim n/2$ and $S_{2n+1}^1 \sim n/2$, $S_{2n}^1 = 0$.

It is clear that

$$\sum_0^\infty |S_n|^p n^{-2} = \infty,$$

but

$$\sum_0^\infty |S_n^1|^p n^{-2-p} < \infty.$$

This shows that replacing S_n by S_n^α is more effective in the study of such integrability problems for a power series.

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Linear, hydrodynamic flow in a rotating saturated porous medium

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Abstract. Linear, steady, axisymmetric flow of a homogeneous fluid in a rigid, bounded, rotating, saturated porous medium is analyzed. The fluid motions are driven by differential rotation of horizontal boundaries. The dynamics of the interior region and vertical boundary layers are investigated as functions of the Ekman number $E(=\nu/\Omega L^2)$ and rotational Darcy number $N(=k\Omega/\nu)$ which measures the ratio between the Coriolis force and the Darcy frictional term. If $N \gg E^{-1/2}$, the permeability is sufficiently high and the flow dynamics are the same as those of the conventional free flow problem with Stewartson's $E^{1/3}$ and $E^{1/4}$ double layer structure. For values of $N \leqslant E^{-1/2}$ the effect of porous medium is felt by the flow; the Taylor-Proudman constraint is no longer valid. For $N \ll E^{-1/3}$ the porous medium strongly affects the flow; viscous side wall layer is absent to the lowest order and the fluid pumped by the Ekman layer returns through a region of thickness $O(N^{-1})$. The intermediate range $E^{-1/3} \ll N \ll E^{-1/2}$ is characterized by double side wall layer structure: (1) $E^{1/3}$ layer to return the mass flux and (ii) $(NE)^{1/2}$ layer to adjust the interior azimuthal velocity to that of the side wall. Spin-up problem is also discussed and it is shown that the steady state is reached quickly in a time scale $O(N)$.

Keywords. Rotation; porous medium; side wall boundary layers.

1. Introduction

The present problem is aimed at understanding the linear, steady, axisymmetric flow of a homogeneous fluid in a rotating porous medium of any permeability. The study of flows through porous media has attracted considerable attention (eg. Lapwood [9], Nield [11], Rudraiah and Srimani [15]) because of its possible and potential applications in petroleum, chemical and nuclear industries, and in geohydrology and geophysical problems. Bretherton and Spiegel [2] suggested in connection with solar spin-down problem that in geophysical or astrophysical flows, the mixed layer near the surface can be modelled as a permeable medium. An extension to this idea has been given by Kroll and Veronis [8] and Howard [7]. Howard considered the effect of side walls also on the spin-up of fluid bounded below by a permeable medium and investigated how the radial flow in the bounded porous bed returns into fluid lying above the porous medium interface. In this problem, the rotational Darcy number

of the side wall boundary layers and determine the manner in which the meridic circulation is closed in a differentially rotating, fluid saturated, bounded por medium. It will be assumed that the solid matrix of the porous medium remains ri It will be shown that for $N \gg E^{-1/2}$ the dynamics of the flow are the same as thos the conventional free flow problem (Stewartson [7]) with Stewartson's $E^{1/3}$ and double layer structure. New results will correspond only to the parametric range $E^{-1/3} \ll N \ll E^{-1/2}$ and (ii) $N \ll E^{-1/3}$ where $E(=\mu/\rho\Omega L^2)$ is the Ekman numbe

The fluid flow through a porous medium is usually described by the Darcy equat

$$\mathbf{V} = \frac{-k}{\mu}(\nabla P - \mathbf{F})$$

where $\mathbf{F}$ is the sum of applied body forces, μ is the viscosity of the fluid and k is permeability of the medium. Here $\mathbf{V}$ is the mean filter velocity rather than the velocity, and the frictional force offered by the solid particles to the fluid flo $-(\mu/k)\mathbf{V}$. Due to the potential nature of the Darcy flow it is not possible to disc many practical problems such as those concerned with instabilities, boundary la and the like. Hence, in the analysis of several instability/boundary layer probl (for e.g., Rudraiah and Musuoka [14], Chakrabarti and Gupta [4], Nield [12, 13 has been found necessary to use the boundary layer type of Brinkman equation which involves viscuous shear in addition to Darcy resistance near the bounda and the usual Darcy equation away from the boundaries. Naturally this equa takes care of the no slip boundary conditions. Brinkman equation may be writter

$$\nabla P - \mathbf{F} = -\frac{\mu}{k}\mathbf{V} + \tilde{\mu}\nabla^2(\mathbf{V})$$

where $\tilde{\mu}$ is an effective viscosity. It is seen from (2) that the Brinkman equation redu to the Navier-Stokes equation as $k \to \infty$ and to the Darcy equation as $k \to 0$. behaviour is to be expected since permeability is in fact a measure of the ease with wh fluids may traverse the medium.

Since the Brinkman model is now known to yield useful results in several f problems through porous media as mentioned earlier, we propose to use it in analysis of the rotating hydrodynamic flow in a porous medium. The problem formulated and non-dimensionalized in § 2. Using Von-Karman similarity transfor tion, the flow between two infinite differentially rotating disks is discussed in § discussion on spin-up is given in § 4. Steady hydrodynamic flow in a rotating conta is discussed in § 5. In § 6 we present our comments and conclusions.

2. Formulation of the problem

We consider a porous medium made of uniform spherical particles and satura with an incompressible fluid of viscosity μ and density ρ. The porous mediun

is disturbed. Departures from a state of rigid body rotation are driven by slow steady axisymmetric motions of the top and bottom walls of the container.

The Navier-Stokes equations governing the steady flow of a homogeneous incompressible fluid in a rotating porous medium based on Brinkman model (see [4]) may be written in a rotating frame of reference as

$$(\mathbf{V}\cdot\nabla)\mathbf{V} + 2\rho\Omega\hat{z} \times \mathbf{V} = -\nabla(P - \tfrac{1}{2}\rho\Omega^2 r^2) - \frac{\mu}{k}\mathbf{V} + \mu\nabla^2\mathbf{V},\ \nabla\cdot\mathbf{V} = 0 \tag{3}$$

where $\mathbf{V}$, $\Omega\hat{z}$, $\hat{z}$, p, ρ, μ, k, and r are, respectively fluid velocity vector, rotation vector, axial unit vector, pressure, density, viscosity, permeability of the medium and radial distance. Following Brinkman who took $\tilde{\mu} = \mu$ and also several other authors (for example refer [4] and [14]) who used the same approximation we replaced $\tilde{\mu}$ in a porous medium by fluid viscosity μ without any loss of generality (for further discussion see Lundgren [10].) The Ekman number now becomes $\mu/\rho\Omega L^2$ instead of $\tilde{\mu}/\rho\Omega L^2$.

The mechanical forcing may be expressed in cylindrical coordinates as

$$\begin{aligned} \mathbf{V} &= \varepsilon\Omega L V_{B,T}(r)\hat{\theta} && \text{at } z = 0,\ L \\ \mathbf{V} &= 0 && \text{at } r = R. \end{aligned}$$

It turns out that the parameter ε is the Rossby number which gives the ratio between the inertial force and the Coriolis force term in (3).

Let us introduce dimensionless variables as follows:

$$\begin{aligned} &\mathbf{V} = \varepsilon\Omega L\mathbf{V}^*,\ p = \tfrac{1}{2}\rho\Omega^2 r^2 + \varepsilon\rho\Omega^2 L^2 p^* \\ &z = LZ^*,\ r = Lr^*. \end{aligned} \tag{4}$$

Assuming that the Rossby number $\varepsilon \ll 1$, we may linearize (3) and write them in dimensionless component form (dropping the asterisks) as

$$-2v = -p_r - \frac{u}{N} + E\left(\nabla^2 - \frac{1}{r^2}\right)u, \tag{5a}$$

$$2u = -\frac{v}{N} + E\left(\nabla^2 - \frac{1}{r^2}\right)v \tag{5b}$$

$$0 = -p_z + E\nabla^2 w - \frac{w}{N}, \tag{5c}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}, \tag{5d}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2},\ \mathbf{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{z},$$

and

$$E = \frac{\mu}{\rho\Omega L^2},\ N = \frac{\rho k\Omega}{\mu}.$$

number. The parameter N is the ratio between the Coriolis force and the Darcy frictional term and it may be called as rotational Darcy number. As in [8] we find it convenient to use parameter N instead of the permeability parameter $P_l[=NE=(k/L^2)]$.

3. Flow between two infinite plates

First we consider the case where the radius of the cylindrical container is infinite. Then we have two infinite discs separated by a distance L. The space in between the disks consists of fluid saturated porous bed made of uniform spherical particles. The simplicity in the present model allows us to write,

$$v = r\,V(z),\; u = r\,U(z)$$

$$p(r,z) = \frac{cr^2}{2} + P(z),\; w = W(z).$$

The resultant ordinary differential equations become

$$-2V = -C - \frac{U}{N} + E\,U_{zz}, \tag{6a}$$

$$2U = -V/N + E\,V_{zz}, \tag{6b}$$

$$0 = -P_z - \frac{W}{N} + E\,W_{zz}, \tag{6c}$$

$$0 = 2U + W_z. \tag{6d}$$

It is easily seen that if we neglect the viscous terms, that is, if we consider only the Darcy model, we cannot satisfy the boundary conditions. Hence, Brinkmann model is appropriate. Adjacent to the bottom and top boundaries we expect Ekman boundary layers modified by Darcy friction. It is a routine matter to obtain the Ekman-Darcy layer structure to lowest order. Writing $z = E^{1/2}\xi$ in the bottom boundary layer, and denoting the boundary layer correction fields by a bar, we get from (6) to lowest order

$$-2\bar{V} = -\frac{\bar{U}}{N} + \bar{U}_{\xi\xi} \tag{7}$$

$$2\bar{U} = -\frac{\bar{V}}{N} + \bar{V}_{\xi\xi} \tag{8}$$

subject to the conditions

$$V_i + \bar{V} = V_B \text{ and } \bar{U} = 0 \quad \text{at } z = 0 \tag{9}$$

where subscript i refers to the interior region. Solving (7) and (8) subject to (9) we get

$$\bar{V} = (V_B - V_i)\exp(-\beta\xi)\cos\gamma\xi \tag{10}$$

$$\bar{U} = (V_B - V_i)\exp(-\beta\xi)\cos\gamma\xi$$

and

$$\bar{W} = \frac{2E^{1/2}}{[illegible]}(V_B - V_i)\exp(-\beta\xi)(\beta\sin [illegible]) \tag{11}$$

where

$$\beta, \gamma = \left(\frac{(1+4N^2)^{1/2} \pm 1}{2N} \right)^{1/2}.$$

It may be noted that $\beta\gamma = 1$ and $\beta^2 - \gamma^2 = 1/N$.

The above analysis reveals that for $N > O(1)$ the Ekman-Darcy layer characterized by a balance between viscous and Coriolis forces in a thickness $O(E^{1/2})$ is dynamically similar to the Ekman layer of the free flow case. The radial mass flux in the layer is $O(E^{1/2})$ and hence by the continuity equation the Ekman suction is $O(E^{1/2})$.

The Darcy frictional term becomes important only when $N = O(1)$ and dominates the Coriolis force term in azimuthal momentum equation (8) for $N < O(1)$.

For $N < O(1)$ it may be seen from (10) or from (7) and (8) that the thickness of the Ekman-Darcy layer is reduced to $O(NE)^{1/2}$. Since the permeability of the medium is considerably less for $N < O(1)$ the radial velocity and hence the radial mass flux in the layer are inhibited and are respectively $O(N)$ and $O(E^{1/2}N^{3/2})$. Hence by continuity of mass flux, the Ekman suction is also inhibited and it becomes $O(E^{1/2}N^{3/2})$.

The top Ekman layer may also be analyzed in a similar way. Since $W_i + \bar{W} = 0$ at $z = 0, 1$ the Ekman layer solutions lead to the compatibility conditions on interior axial velocity. These may be written for W_i or ψ_i as

$$W_i = \pm \frac{2E^{1/2}\gamma}{\beta^2 + \gamma^2}(V_i - V_{B,T}) \quad \text{at } z = 0,1 \tag{12}$$

$$\psi_i = \mp \frac{E^{1/2}\gamma}{\beta^2 + \gamma^2}(v_i - v_{B,T}) \quad \text{at } z = 0, 1$$

where ψ is the stream function defined as

$$\frac{\partial \psi}{\partial z} = u \text{ and } -\frac{1}{r}\frac{\partial}{\partial r}(r\psi) = w.$$

From (4) we obtain in the interior region

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}(W_i) = 0. \tag{13}$$

From (6), (12) and (13) we get

$$\begin{aligned} W_i &= Az + B \\ U_i &= -A/2 \end{aligned} \tag{14}$$

$$V_i = \frac{\gamma E^{1/2}}{\beta^2 + \gamma^2} \frac{1}{\left(\frac{1}{N} + \frac{4E^{1/2}\gamma}{\beta^2 + \gamma^2} \right)} 2(V_B + V_T)$$

and

$$B = -\frac{2E^{1/2}\gamma}{\beta^2+\gamma^2}(V_B - V_i). \tag{16}$$

It is easily seen that for $N \gg E^{-1/2}$ the solutions (14) and (15) reduce to the well-known solutions corresponding to conventional free flow situation. As N decreases to $O(E^{-1/2})$, the Darcy friction becomes important in the interior region and hence the Taylor-Proudman constraint is no longer valid in this region. The axial flow becomes z-dependent, and hence radial flow occurs in the interior region. Both the variables u and w are $O(E^{1/2})$ for $1 \leqslant N \leqslant E^{-1/2}$ while the azimuthal flow is $O(NE^{1/2})$. Even though the Ekman suction is still $O(E^{1/2})$, the motion of the end boundaries is not communicated to the interior fluid as the angular momentum of the interior region can no longer be treated as conserved. Since $V_i < O(1)$, there can be Ekman section of $O(E^{1/2})$ only at the lower boundary if $V_T = 0$.

The interior solutions for $N \ll E^{-1/2}$ will be presented here for $V_T = 0$ and these solutions will be made use of while investigating the side wall boundary layer structure. For $V_T = 0$ we get from (14)–(16),

$$\begin{aligned} V_i &= \frac{2NE^{1/2}\gamma}{\beta^2+\gamma^2}V_B \\ U_i &= -\frac{A}{2} = -\frac{\gamma E^{1/2}}{\beta^2+\gamma^2}V_B \\ W_i &= \frac{2E^{1/2}\gamma}{\beta^2+\gamma^2}V_B(z-1) \\ \psi_i &= -\frac{E^{1/2}\gamma}{\beta^2+\gamma^2}r(z-1). \end{aligned} \tag{17}$$

For $N < O(E^{-1/2})$ Ekman layer solutions to lowest order at the bottom plate can be obtained from (10) directly. They are

$$\begin{aligned} \bar{V} &= \exp(-\beta\xi)\, V_B \cos\gamma\xi \\ \bar{U} &= \exp(-\beta\xi)\, V_B \sin\gamma\xi. \end{aligned} \tag{18}$$

For $N < O(1)$, these solutions may be written as $\bar{V} \simeq V_B \exp(-\bar{\xi})$ and $\bar{U} \simeq V_B \bar{\xi} \exp(-\bar{\xi})$ where $\bar{\xi} = N^{-1/2}\xi$.

4. Unsteady flow

Before proceeding to the analysis of side wall boundary layers, we shall present a brief discussion of the spin-up problem. This problem has been discussed earlier by

established on the discs in a time of order unity. Hence we may apply the Ekman compatability conditions (12) to unsteady motions whose time scale exceeds $O(1)$.

The azimuthal momentum equation governing the interior is

$$\frac{\partial V}{\partial t} + 2U = -\frac{V}{N}. \tag{19}$$

It may be noted that for $N \gg E^{-1/2}$ Darcy term is unimportant and for $N \leqslant E^{-1/2}$, all the three terms are of the same order in a time scale of order N. It is seen from (17) that $U \neq f(z)$ in the interior. Hence the continuity equation (6d) gives

$$W = -2Uz + f(r, t).$$

The Ekman layers may be considered as quasi-steady during the spin-up period. Hence, applying the Ekman compatibility conditions (12) and eliminating $f(r, t)$ we get

$$V_t + \frac{4E^{1/2}\gamma}{\beta^2+\gamma^2}(V-1) = -\frac{V}{N}$$

whose solution with $V = 0$ at $t = 0$ is

$$V = \frac{4E^{1/2}\gamma}{(\beta^2+\gamma^2)\left(\frac{4E^{1/2}\gamma}{\beta^2+\gamma^2}+\frac{1}{\mathrm{N}}\right)}\left[1-\exp\left(-\left(\frac{1}{\mathrm{N}}+\frac{4E^{1/2}\gamma}{\beta^2+\gamma^2}\right)\right)t\right]. \tag{20}$$

Thus the fluid in the interior reaches a steady state in a time of order

$$\left(\frac{1}{\mathrm{N}}+\frac{4E^{1/2}}{\beta^2+\gamma^2}\right)^{-1}.$$

For $N \gg E^{-1/2}$, the solution for V reduces to that of the conventional rotating, non-porous, free flow problem (for e.g., see Greenspan (15)). However as N takes values less than $O(E^{-1/2})$ the interior region can no longer be treated as inviscid. The Darcy frictional term in (19) becomes important and the steady state is reached quickly in a time $O(N)$. The fluid in the interior rotates uniformly, but, at a speed which is less than $O(1)$. The steady state azimuthal velocity for $1 \leqslant N \leqslant E^{-1/2}$ is $O(NE^{1/2})$ and it is $O(N^{5/2}E^{1/2})$ for $N \ll 1$. The absence of zonal velocity of $O(1)$ in the interior may be attributed to the fact, that the angular momentum is no longer conserved in the porous medium case because of the presence of the Darcy term $(\mu/K)\mathbf{V}$ in the equation of motion.

From (19) it may be seen that even if $V = O(1)$ initially, it would decay exponentially with time.

5. Flow in a container—Side wall boundary layers

$\Omega(1+\varepsilon)$ leading to $V_B = 1$. The upper boundary and the lateral wall of the cylindrical container are at rest in the rotating frame.

Since $\partial/\partial r \gg \partial/\partial z$ near the side walls, we may obtain from (5) a single equation for a single field variable say v, as

$$E^2\frac{\partial^6 v}{\partial r^6} + E^2\frac{\partial^6 v}{\partial r^4 \partial z^2} - 2EN^{-1}\left(\frac{\partial^4 v}{\partial r^4} + \frac{\partial^4 v}{\partial r^2 \partial z^2}\right) \tag{21}$$

(1) (2) (3) (4)

$$+N^{-2}\left(\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial z^2}\right) + 4\frac{\partial^2 v}{\partial z^2} = 0.$$

(5) (6)

It is already known that for $N \gg E^{-1/2}$, the flow structure is the same as that of the conventional non-porous free flow problem where the outer Stewartson layer has a $O(E^{1/4})$ and the inner one has a thickness $O(E^{1/3})$.

If the first term in (21) is comparable to the sixth term we get Stewartson's $E^{1/3}$ layer. From (21) it is easily seen that this force balance is valid for $N \gg E^{-1/3}$. The thickness of the modified outer Stewartson layer cannot be directly obtained from the equation. However, it is known (Barcilon [1]) that the outer Stewartson layer has a thickness equal to the distance through which vorticity can diffuse in spin-up time. Since the spin-up time for our problem is $O(N)$, the modified thickness of the outer Stewartson layer turns out to be $O(E^{1/2}N^{1/2})$.

As $N \to E^{-1/3}$, it is seen that the outer Stewartson layer merges with the $E^{1/3}$ layer. First and third terms in (21) can balance each other in a layer of width $E^{1/2}N^{1/2}$. This force balance is possible if $N \ll E^{-1/3}$. Further fifth and sixth terms can balance each other to give rise to another layer of width N^{-1} if $N \ll E^{-1/3}$. Thus, there can exist N^{-1} and $E^{1/2}N^{1/2}$ layers in the parameter range $1 \ll N \ll E^{-1/3}$.

As $N \to O(1)$, the N^{-1} layer merges with the interior. We may have only a single side wall boundary layer of width $E^{1/2}N^{1/2}$ for $N \leqslant O(1)$. It may be noted that the thickness of the possible side wall layer is equal to the Ekman layer thickness for $N \ll 1$. (It will be seen that this $E^{1/2}N^{1/2}$ layer does not exist to the lowest order for $N \ll E^{-1/3}$).

Thus the problem has four parametric ranges: (i) $N \gg E^{-1/2}$ (ii) $E^{-1/3} \ll N \ll E^{-1/2}$. (iii) $1 \ll N \ll E^{-1/3}$ and (iv) $N \leqslant O(1)$. As mentioned earlier, for $N \gg E^{-1/2}$ the results correspond to free flow problem analyzed by Stewartson. The Darcy term becomes important only for $N \ll E^{-1/2}$ and hence new results correspond only to this range. A unified analysis could be given for the parameter range $N \ll E^{-1/3}$.

It is clear that for $E^{-1/3} \ll N \ll E^{-1/2}$ the two possible side wall layers are Stewartson type. The physical properties of these layers have been described by Barcilon [1]. If the interior azimuthal velocity

$$V_i(z) = \langle V \rangle + V(z)$$

where

one Ekman layer to the other. The properties of the boundary layers will help u scale the correction variables in the respective layers and obtain their govern equations. For example, the correction term for the azimuthal velocity in the $(NE$ layer should be expected to be $O(NE^{1/2})$ to adjust the interior azimuthal velo $O(NE^{1/2})$ to that at the side wall. The orders of magnitude of other variables then be found from (5). The $E^{1/3}$ layer is required to support the mass flux $O(E$ pumped by the Ekman layer. Hence the vertical velocity in this layer should be $O(E$

5.2 *Solution for* $E^{-1/3} \ll N \ll E^{-1/2}$

The solutions in the interior region are (see § 3, eq. (17))

$$v_i = NE^{1/2} r,\ u_i = \frac{-E^{1/2} r}{2},\ w_i = E^{1/2}(z-1),\ \psi_i = -E^{1/2}\frac{r}{2}(z-1).$$

5.2.1. $(NE)^{1/2}$ *layer* starting with the thickest layer, let us introduce a stretc variable η defined as

$$\eta = (a-r)(NE)^{-1/2}$$

and let us scale the various fields which are corrections to the interior fields as

$$v = NE^{1/2}\tilde{v},\ u = E^{1/2}\tilde{u},\ w = N^{-1/2}\tilde{w},\ p = EN^{3/2}\tilde{p}.$$

To the lowest order the equations of motion become (see eqs (5)),

$$\begin{aligned} -2\tilde{v} &= \tilde{P}_\eta \\ 2\tilde{u} &= -\tilde{v} + \tilde{v}_{\eta\eta} \\ \tilde{P}_z &= 0 \\ \tilde{u}_n &= \tilde{w}_z. \end{aligned}$$

The Ekman compatibility conditions (12a) yield

$$\tilde{w} = 0 \text{ at } z = 0, 1.$$

Since $\tilde{w}_{zz} = 0$ in this layer, we should have $\tilde{w} \equiv 0$ to satify the conditions (21). H $\tilde{u} \equiv 0$.

Since this layer is expected to adjust the interior azimuthal velocity to zero a side wall

$$\tilde{v} = -a \text{ at } \eta = 0.$$

We should have from (20b)

$$\tilde{v} = -a\exp(-\eta).$$

the boundary layer equations are

$$-2\hat{v} = -\hat{p}_\mu,\ 2\hat{u} = \hat{v}_{\mu\mu}$$
$$\hat{p}_z = \hat{w}_{\mu\mu},\ \hat{u}_\mu = \hat{w}_z. \qquad (2$$

Where the correction fields have been scaled the following way

$$u = E^{1/2}\hat{u},\ v = E^{1/6}\hat{v},\ w = E^{1/6}\hat{w},\ p = E^{1/2}\hat{p}.$$

From (23) we get

$$\frac{\partial^6 \hat{w}}{\partial \mu^6} + \frac{\partial^2 \hat{w}}{\partial z^2} = 0. \qquad (2$$

Since the $E^{1/3}$ layer is thicker than the Ekman layer the compatibility conditions (1
can be used to get

$$\hat{w} = 0 \quad \text{at } z = 0, 1. \qquad (2$$

The boundary conditions at $\mu = 0$ are

$$\hat{w} = 0$$
$$\hat{v} = 0 \text{ and } u_i + \hat{u} = 0. \qquad (2$$

The boundary condition $v = 0$ at $\mu = 0$ is equivalent to

$$\hat{w}_{\mu\mu\mu} = 0 \quad \text{at } \mu = 0. \qquad (2$$

While the boundary condition on u may be written as

$$\frac{a}{2}w_i + \int_0^\infty \hat{w}\, d\mu = 0 \qquad (2$$

where $w_i = (z-1)$ from (17). Solution of (24) subject to boundary conditions (25)–(28

$$\hat{w} = \sum_{n=1}^{\infty} \frac{2\gamma_n a}{n\pi\sqrt{3}} \exp - (\gamma_n/2) \sin\left(\frac{\sqrt{3}\gamma_n \mu}{2}\right) \sin n\pi z$$

where $\gamma_n = (2n\pi)^{1/3}$.

It may be noted that for $N \gg E^{-1/2}$, that is in the case of conventional free fl
problem the outer Stewartson layer also supports the mass flux. However, in t
present porous medium case where $N \ll E^{-1/2}$, it is seen from the solutions of (NE)
layer that this layer is weak and does not support the mass flux $O(E^{1/2})$. The physi
reason is obvious; the azimuthal velocity in this layer is less than order unity. T
Ekman compatibility conditions (12) reveal that this layer cannot support a m
flux $O(E^{1/2})$. The interior mass flux is closed only through the $E^{1/3}$ layer. T
circulation pattern is shown in figure 1.

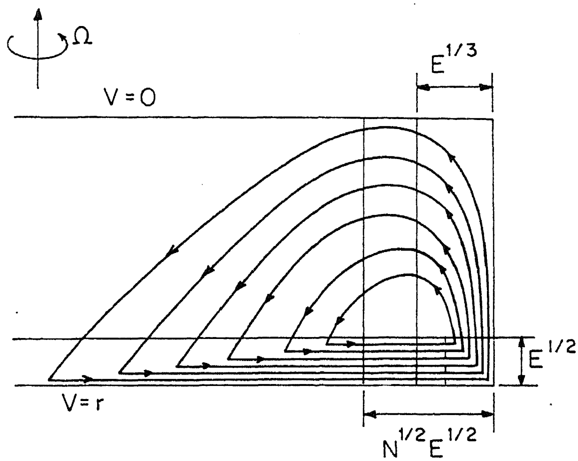


Figure 1. Schematic diagram showing meridional circulation in (r, z) plane for por parameter $E^{-1/3} \ll N \ll E^{-1/2}$.

5.3 *Solution for* $1 \ll N \ll E^{-1/3}$

As mentioned in § 5.1 there can exist two side wall layers in this range: (i) $E^{1/2}$ layer and (ii) N^{-1} layer. The governing equations for $E^{1/2}N^{1/2}$ layer remain the s as those in the parametric range $E^{-1/3} \ll N \ll E^{-1/2}$ and the correction field to zonal velocity can at most be $O(NE^{1/2})$. Hence, this layer, as before, cannot sup the mass flux pumped by the Ekman layer. Also, it will be seen that this layer is necessary even to adjust the interior azimuthal velocity to zero at the side wall. new boundary layer of thickness $O(N^{-1})$ helps in closing the meridional circula of fluid and satisfy the boundary condition on zonal velocity.

5.3.1 *1/N layer* We expect this layer to carry the mass flux $O(E^{1/2})$. Using do bar to denote the various correction fields within this boundary layer and defi a stretched variable,

$$\sigma = N(a - r)$$

the boundary layer equations are

$$-2\bar{v} = \bar{p}_\sigma;\ 2\bar{u} = -\bar{v}$$

$$\bar{p}_z = -\bar{w};\ \bar{u}_\sigma = \bar{w}_z,$$

where the correction fields have been scaled in the following way:

$$u = E^{1/2}\bar{u},\ v = NE^{1/2}\bar{v},\ p = E^{1/2}\bar{p},\ w = NE^{1/2}\bar{w}.$$

In terms of the stream function ψ defined earlier the governing equation for ψ in

layer is

$$\bar{\psi}_{\sigma\sigma} + 4\bar{\psi}_{zz} = 0$$

subject to boundary conditions

$$\bar{\psi} = 0 \quad \text{at } z = 0, 1$$

and

$$\bar{\psi} = a(z-1)/2 \quad \text{at } \sigma = 0.$$

Solving (30) subject to (31) we get

$$\bar{\psi} = -\sum_{n=1}^{\infty} \frac{a}{n\pi} \exp(-2n\pi\sigma) \sin(n\pi z).$$

Solutions for the other variables may be obtained from (32) and (29). However solutions may not be uniformly convergent or convergent near the side wall.

It will now be shown that the $E^{1/2}N^{1/2}$ layer does not exist to lowest orde

Since $u_i + \bar{u} = 0$ at $r = a$, we should have from (17) $\bar{u} = a/2$ at $r = a$.

From (29) we get $\bar{v} = -a$ at $r = a$. Since $(v_i + \bar{v})$ vanishes at the side wall, th layer is itself sufficient to bring the azimuthal velocity to zero at the side wall

It may be borne in mind, that, as $w_i + \bar{w}$ does not vanish at $r = a$ the E^1 layer can exist to higher order.

The meridional flow pattern is similar to that shown in figure 1. The mass flu flowing out radially in the bottom Ekman layer spreads out from the corner and returns through the thicker region of width $O(N^{-1})$.

5.4 *Solution for* $N \leqslant O(1)$

The N^{-1} region merges with the interior for $N = O(1)$ and the mass flux pum the Ekman layer returns through the interior. It may be expected that the E^1 layer will disappear to lowest order. Since the governing equations of this la $N \leqslant O(1)$ differ from (20), we give below the equations and show that this laye not occur.

5.4.1 $E^{1/2}N^{1/2}$ *layer.*

From (17) we have

$$u_i = O(E^{1/2}N^{3/2}),\ w_i = O(E^{1/2}N^{3/2}) \text{ and } v_i = O(E^{1/2}N^{5/2}).$$

In view of the magnitudes of the interior variables, we may scale the varia this layer as

$$v = E^{1/2}N^{5/2}\tilde{v},\ w = N\tilde{w},\ u = E^{1/2}N^{3/2}\tilde{u},\ p = NE\tilde{p}.$$

The equations in the boundary layer become

where

$$\eta=(NE)^{-1/2}(a-r).$$

Since $\tilde{w}=0$ at $\eta=0$ we have from (33c) $\tilde{w}\equiv 0$. Hence from (33d) $\tilde{u}\equiv 0$.

Since the boundary condition on v will be satisfied by the interior region itsel get $\tilde{v}=0$ from (33b). Hence, it is to be concluded that this layer does not exis lowest order.

5.4.2 *Interior region.* Neglecting viscous terms, the equation for the stream func in the interior becomes,

$$\psi_{\gamma\gamma}+\frac{1}{r}\psi_r-\frac{\psi}{\gamma^2}+(1+4N^2)\psi_{zz}=0.$$

Since the side wall layer cannot support the mass flux the interior itself should sa the boundary condition on ψ at the side wall.

Hence,

$$\psi=0 \quad \text{at } r=a.$$

From (12) the compatibility conditions on ψ give

$$\psi=v_B=r \quad \text{at } z=0$$

$$\psi=-v_T=0 \quad \text{at } z=1.$$

Solution of (34) subject to boundary conditions (35) and (36) is

$$\psi=(1-z)r-2\sum_{n=1}^{\infty}\frac{I_1(\lambda_n r)}{I_1(\lambda_n)}\sin n\pi z.$$

Where I_1 represents modified Bessel function of first order, and

$$\lambda_n=n\pi\sqrt{1+4N^2}.$$

Since $I_1(X)\sim(1/(2\pi X)^{1/2})\exp(X)$ for $X\gg 1$, it is easily seen from (37) that for $N\gg 1$ second term corresponds to the solution (32) obtained earlier in the N^{-1} layer ir parametric range $1\ll N\ll E^{-1/3}$.

6. Summary, comments and conclusions

The present analysis provides a unified picture of the linear dynamics of rota homogeneous fluids in a porous medium. The foregoing analysis reveals that critical values for rotational Darcy number $N(=k\Omega/\nu)$ arise and divide $(E,N)-$s into three distinct regions. For $N\gg E^{-1/2}$, that is, when the medium is hi permeable the flow dynamics in all regions reduce to that of free fluid flow. Th (i) Taylor-Proudman theorem is valid in the interior and hence the flow structu this region is z-independent. (ii) Stewartson boundary layers of thicknesses $E^{1/3}$

return through the side walls, especially the $E^{1/3}$ layer and (iii) Interior is controlled by Ekman suction.

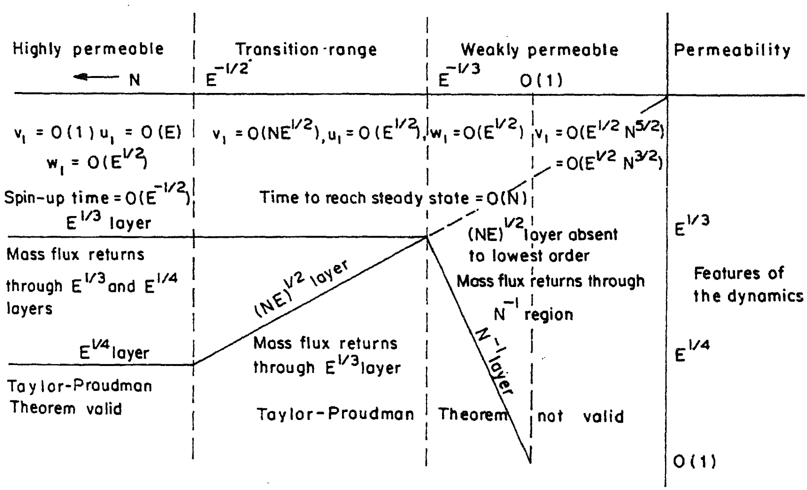
As N decreases below $E^{-1/2}$, the effect of the porous medium is felt by the flow. The Darcy term representing the frictional forces offered by the solid particles in the medium becomes important. The angular momentum is no longer conserved and hence the interior is not spun up though the spin-up time is reduced to $O(N)$. Ekman suction loses control over the interior and the interior azimuthal velocity is less than $O(1)$. Taylor-Proudman constraint is broken and the vertical velocity varies linearly with z. As a result there is a radial flow in the interior. For $E^{-1/3} \ll N \ll E^{-1/2}$, the side wall boundary layer structure still consists of the two Stewartson layers, but the outer layer is modified to have a thickness $O(N^{1/2}E^{1/2})$. Since the correction field variable for the azimuthal velocity in the outer layer is less than order unity, this layer cannot partake in supporting the mass flux. The $E^{1/3}$ layer alone takes part in closing the meridional flow. As $N \to E^{-1/3}$, the $(NE)^{1/2}$ layer merges with the $E^{1/3}$ layer.

For $1 \ll N \ll E^{-1/3}$ the return mass flux is confined to region of width N^{-1}. This region spreads out and merges with the interior as $N \to O(1)$. In fact, as N reduces, that is as the medium becomes less permeable, the medium offers more resistance and so the mass flux cannot be returned through a thin layer. The return mass flux spreads out and passes through interior region. This is made possible because the rotational constraint is broken and variations with axial coordinate are allowed.

Further, for $N \ll E^{-1/3}$ the boundary condition on azimuthal velocity is satisfied by the interior region itself. As a result, the $E^{1/2}N^{1/2}$ viscous layer which in principle can exist for $N \ll E^{-1/3}$ does not exist to lowest order. It can now exist to higher order to adjust the vertical velocity in the interior region to zero at the side wall.

The above conclusions are summarized in figure 2, in which the modifications of the various features of the dynamics are schematically represented as the rotational Darcy number N changes.

The non-dimensional parameter N could be of order unity in several practical cases and in such cases the side wall boundary layer structure may be unimportant. Finally,



we wish to mention that the same problem may be understood by considering a disc configuration and using integral transform methods [16].

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On the $|\bar{N}, p_n|_k$ summability factors for infinite series

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Abstract. In this paper a theorem on $|\bar{N}, p_n|_k$ summability factors of infinite series, which generalizes a theorem of Bor [2], has been proved.

Keywords. Summability factors; infinite series.

1. Introduction

Let Σa_n be a given infinite series with the sequences of partial sums (s_n). Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geqslant 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series Σa_n is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{3}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability. The series Σa_n is said to be bounded $[N, p_n]_k$, $k \geqslant 1$, if (see [1])

$$\sum_{v=1}^{n} p_v |s_v|^k = O(P_n) \text{ as } n \to \infty. \tag{4}$$

If we take $k = 1$, then $[\bar{N}, p_n]_k$ boundedness is the same as $[\bar{N}, p_n]$ boundedness.

2. Quite recently Bor[2] proved the following theorem.

Theorem A. *Let Σa_n be bounded $[\bar{N}, p_n]$. Let (p_n) be a positive sequence such that $P_n \to \infty$ as $n \to \infty$ and*

$$\frac{1}{n} = O(p_n). \tag{5}$$

Suppose there are sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leqslant \beta_n \tag{6}$$

$$\beta_n \to 0 \text{ as } n \to \infty \tag{7}$$

$$\sum_{n=1}^{\infty} nP_n|\Delta\beta_n| < \infty \tag{8}$$

$$P_n|\lambda_n| = O(1) \text{ as } n \to \infty. \tag{9}$$

Then the series $\Sigma a_n \lambda_n$ is summable $|\bar{N}, p_n|$.

3. The object of the present paper is to generalize Theorem A for $|\bar{N}, p_n|_k$, with $k \geqslant 1$, by proving the following theorem.

Theorem. *Let Σa_n be bounded $[\bar{N}, p_n]_k$. If the sequences (p_n), (β_n) and (λ_n) such that conditions (5)–(9) of Theorem A are satisfied, then the series $\Sigma a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.*

It should be noted that if we take $k = 1$ in this theorem, then we get Theorem A.

4. We need the following lemma for the proof of our theorem.

Lemma 2. Under the conditions on (P_n) and (β_n) as taken in statement of the theorem, the following conditions hold, when (8) is satisfied.

$$nP_n\beta_n = O(1) \text{ as } n \to \infty \tag{10}$$

$$\sum_{n=1}^{\infty} P_n\beta_n < \infty. \tag{11}$$

5. Proof of the theorem

Let (T_n) be the sequence of the $(\bar{N}, p_n)$ mean of the series $\Sigma a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v. \tag{12}$$

Then, for $n \geqslant 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v. \tag{13}$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v) s_v + \frac{1}{P_n} p_n s_n \lambda_n \\ &= \frac{1}{P_n} p_n s_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v s_v \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say.} \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3. \tag{14}$$

Since $|\lambda_n| = O(1/P_n) = O(1)$, by (9), we have

$$\begin{aligned} \sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,1}|^k &= \sum_{n=1}^{m} |\lambda_n|^k p_n |s_n|^k \frac{1}{P_n} = \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| p_n |s_n|^k \frac{1}{P_n} \\ &= O(1) \sum_{n=1}^{m} |\lambda_n| p_n |s_n|^k = O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{r=1}^{n} p_r |s_r|^k \\ &\quad + O(1) |\lambda_m| \sum_{n=1}^{m} p_n |s_n|^k \\ &= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| P_n + O(1) |\lambda_m| P_m = O(1) \sum_{n=1}^{m-1} \beta_n P_n \\ &\quad + O(1) |\lambda_m| P_m = O(1) \end{aligned}$$

as $m \to \infty$, by virtue of the hypotheses of the theorem and Lemma.

Now, applying Hölder's inequality, we have that

$$\begin{aligned} \sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,2}|^k &= \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^{n-1} p_v \lambda_v s_v \right|^k \\ &\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |\lambda_v|^k |s_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^{k-1} |\lambda_v| |s_v|^k \frac{1}{P_v} \\ &= O(1) \sum_{v=1}^{m} |\lambda_v| p_v |s_v|^k. \end{aligned}$$

As in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,2}|^k = O(1) \sum_{v=1}^{m} |\lambda_v| p_v |s_v|^k = O(1) \text{ as } m \to \infty.$$

Finally, using the fact that $1/v = O(p_v)$, by (5), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,3}|^k &= \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^{n-1} P_v \Delta\lambda_v s_v \right|^k \\
&\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| |s_v| \right\}^k \\
&\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \beta_v |s_v| \right\}^k \\
&\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v \beta_v |s_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \beta_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} P_v \beta_v |s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \beta_v |s_v|^k = O(1) \sum_{v=1}^{m} \frac{1}{v} P_v \beta_v |s_v|^k \\
&= O(1) \sum_{v=1}^{m} v \beta_v p_v |s_v|^k = O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) P_v + O(1) m \beta_m P_m \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| P_v + O(1) \sum_{v=1}^{m-1} \beta_{v+1} P_v \\
&\quad + O(1) m \beta_m P_m = O(1) \text{ as } m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma.

Therefore, we get

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,r}|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3.$$

This completes the proof of the theorem.

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Erratum

(*Proc. Math. Sci.* **98** 25 (1988))

"On a class of slowly changing function"

by ANAND PRAKASH SINGH

Replace line 10 from below to line 4 from below on p. 26 by the following:

And so

$$-\varepsilon\int_a^k \frac{1}{t}\mathrm{d}t < \int_a^k \frac{r^t \log r J'(r^t)}{J(r^t)}\mathrm{d}t < \varepsilon\int_a^k \frac{1}{t}\mathrm{d}t.$$

Thus

$$-\varepsilon\log\frac{k}{a} < \log\frac{J(r^k)}{J(r^a)} < \varepsilon\log\frac{k}{a}.$$

And so

$$-\varepsilon\log\frac{b}{a} < \log\frac{J(r^k)}{J(r^a)} < \varepsilon\log\frac{b}{a}.$$

It now easily follows that

Complete positivity, tensor products and C*-nuclearity for inverse limits of C*-algebras

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Abstract. The paper aims at developing a theory of nuclear (in the topological alge sense) pro-C^*-algebras (which are inverse limits of C^*-algebras) by investigating comp positive maps and tensor products. By using the structure of matrix algebras o pro-C^*-algebra, it is shown that a unital continuous linear map between pro-C^*-alg A and B is completely positive iff by restriction, it defines a completely positive map bet the C^*-algebras $b(A)$ and $b(B)$ consisting of all bounded elements of A and B. In the metri case, A and B are homeomorphically isomorphic iff they are matricially order isomo The injective pro-C^*-topology α and the projective pro-C^*-topology ν on $A \otimes B$ are s to be minimal and maximal pro-C^*-topologies; and α coincides with the topolo biequicontinous convergence iff either A or B is abelian. A nuclear pro-C^*-algebra A i that satisfies, for any pro-C^*-algebra (or a C^*-algebra) B, any of the equivalent requiren (i) $\alpha = \nu$ on $A \otimes B$ (ii) A is inverse limit of nuclear C^*-algebras (iii) there is only one admi pro-C^*-topology on $A \otimes B$ (iv) the bounded part $b(A)$ of A is a nuclear C^*-algebra (v continuous complete state map $A \to B^*$ can be approximated in simple weak* conver by certain finite rank complete state maps. This is used to investigate permanence prop of nuclear pro-C^*-algebras pertaining to subalgebras, quotients and projective and ind limits. A nuclearity criterion for multiplier algebras (in particular, the multiplier alge Pedersen ideal of a C^*-algebra) is developed and the connection of this C^*-algebraic nucl with Grothendieck's linear topological nuclearity is examined. A σ-C^*-algebra A is a n space iff it is an inverse limit of finite dimensional C^*-algebras; and if abelian, then isomorphic to the algebra (pointwise operations) of all scalar sequences.

Keywords. Inverse limits of C^*-algebras; completely positive maps; tensor products; n C^*- and nuclear pro-C^*-algebras; multiplier algebras; nuclear space.

1. Introduction and Preliminaries

A *topological *algebra A* is an involutive linear associative algebra (with identi over complex scalars admitting a Hausdorff topology such that A is a topolo vector space in which the multiplication and the involution are continuou *pro-C^*-algebra* is a complete topological *algebra A the topology on whic determined by the collection $S(A)$ of all continuous C^*-seminorms on it; equivale A is homeomorphically *isomorphic to an inverse limit of C^*-algebras. A *σ-C^*-alg* is a metrizable pro-C^*-algebra. Besides an intrinsic interest in pro-C^*-algebr topological algebras ([1], [11], [13], [14], [19], [20] and references therein), it has shown recently that they provide an important tool in investigation of certain as of C^*-algebras (like multipliers of the Pedersen ideal, tangent algebra of a C^*-alg

[24], pseudodifferential operators [8] and quantum field theory [9]. In the literat inverse limits of C^*-algebras have been given different names such as b^*-algeb m-convex-C^*-algebras or LMC*-algebras; the more appropriate pro(jective limit C^*-algebras is a recent suggestion [23], [24] following [28].

The present paper aims at developing a theory of nuclear pro-C^*-algebras; this requires investigating tensor products of such algebras and complete positi of linear maps. The significance of this has been noted in [23, p. 175]. Ur C^*-algebras and locally convex spaces, there are at least two concepts of nuclea for pro-C^*-algebras, viz. nuclearity in topological algebraic sense (an extensio C^*-nuclearity [5], [6], [17]), and Grothendieck's linear topological nuclea [26, Chapter III, § 7]. Except for a final remark, we mean the former.

Given a pro-C^*-algebra A, each $p \in S(A)$ determines a C^*-algebra $A_p = A/N_p$ (N $\{x \in A | x_p = 0\}$, a *ideal in A), with C^*-norm $\|x_p\|_p = p(x)$ where $x_p = x + N_p$. point is that $(A_p, \|\cdot\|_p)$ is complete [1], [27]; and A admits an inverse l decomposition $A = \varprojlim_{p \in \Delta} A_p$, Δ being a confinal subset of $S(A)$. The *bounded pa* A is the *subalgebra $b(A) = \{x \in A | \sup_{p \in S(A)} p(x) < \infty\}$, a C^*-algebra with n $\|x\|_\infty = \sup_{p \in S(A)} p(x)$ continuously embedded in A [23]. A crucial fact abou pro-C^*-algebra is that $b(A)$ is dense in A [1], [23]; and this, in fact, characte pro-C^*-algebras [2]. Let $M_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ matrices. G pro-C^*-algebras A and B, a linear map $\phi: A \to B$ is *completely positive* if for eac $\phi_n = \phi \otimes id: A \otimes M_n(\mathbb{C}) = M_n(A) \to B \otimes M_n(\mathbb{C}) = M_n(B)$ is positive. In § 2, by obtai a pro-C^*-analogue of Stinespring's Theorem, it is shown that a continuous completely positive iff $\phi(b(A)) \subset b(B)$ and $\phi: b(A) \to b(B)$ is a completely positive between C^*-algebras. This is used to show that σ-C^*-algebras A and B are homeor phically *isomorphic iff they are matricially order isomorphic. In § 3, the te product of pro-C^*-algebras A and B is investigated. Two standard pro-C^*-topolo on $A \otimes B$ are introduced in [13], viz. the *injective tensorial pro-C^*-topolog* and the *projective tensorial pro-C^*-topology* ν, which correspond respectively to least C^*-norm $\|\cdot\|_{\min}$ and the greatest C^*-norm $\|\cdot\|_{\max}$ in case of C^*-alge [27, Chapter IV]. It is shown in Theorem 3.2 that any admissible pro-C^*-topo τ (in the sense of [13, Definition 2.1]) on $A \otimes B$ satisfies $\alpha \leqslant \tau \leqslant \nu$; and if either B is abelian, then $\alpha = \tau = \nu = \varepsilon$, ε being the topology of biequicontinuous converge (In fact, a slightly stronger result is proved, see Remark 3.3). This impr [13, Proposition 3.1], wherein the conclusion $\tau \leqslant \nu$ has been obtained under additional assumptions that both A and B are metrizable and the completion A is a Q-algebra (i.e., the invertible elements form an open set). The other half of Theorem also improves [13, Theorem 3.1] wherein the conclusion $\alpha = \varepsilon$ is obta under the assumptions that A and B are Q-algebras one being barreled and other commutative. In fact, a pro-C^*-algebra that is a Q-algebra is a C^*-alg [23, Proposition 1.14]; and hence the relevant results in [13] dealing with Q- C^*-algebras are just the usual C^*-algebra results. Thus our result provides a com analogue of [27, Chapter IV, Theorem 4.19] modulo the problem whether pro-C^*-topology on $A \otimes B$ is necessarily admissible. Further, it is shown in Theoren that the above conclusion $\varepsilon = \alpha = \tau = \nu$ characterizes commutativity of either A This extends [27, Chapter IV, Theorem 4.14]. Continuous states on $A \hat{\otimes}_\nu B$ are sh to correspond to continuous completely positive maps $A \to B^*$(= the dual). machinery developed in § 2 and § 3 is used to develop a theory of nuclear pro-C^*-alge

in §4. Following [23, p. 175], a pro-C^*-algebra A is called *nuclear* if for each $p \in S$ the C^*-algebra A_p is nuclear in the sense [17] that for any C^*-algebra B, $\|\cdot\|_{\min} = \|\cdot$ on $A_p \otimes B$. It is shown that A is nuclear iff A is an inverse limit of nuclear C^*-alge (with the maps of the inverse system assumed surjective) iff for any pro-C^*-alg (respectively, C^*-algebra) B, there is only one admissible pro-C^*-topology on A A basic result in C^*-theory is that a C^*-algebra A is nuclear iff A^{**} (the second d is semidiscrete. Using this via the universal representation, it is shown in the n Theorem 4.5 that a pro-C^*-algebra A is nuclear iff $b(A)$ is a nuclear C^*-algebra. is used to investigate permanence properties of nuclear pro-C^*-algebras pertai to hereditary subalgebras, quotients and products, as well as projective and induc limits. By using the structure of continuous state space of an m-convex *algebra it is shown in Theorem 4.11 that if A is nuclear, then for any pro-C^*-algebra B, continuous complete state map $A \to B^*$ can be approximated in simple we convergence by continuous complete state maps of finite ranks; and a weaker conv holds. This is a pro-C^*-analogue of a basic result of Lance [16]. To illustrate application, in §5 a nuclearity criterion for the multiplier algebra of the Pede ideal of a C^*-algebra [18] is obtained. For this, it is shown that for a pro-C^*-algebr its multiplier algebra $M(A)$ [23] is nuclear pro-C^*-algebra iff $b(A)$ and the general Calkin algebra $M(b(A))/b(A)$ are nuclear C^*-algebras. Finally, we examine interrelation between Grothendieck's linear topological nuclearity and nuclearit the sense of the present paper. In fact, a σ-C^*-algebra A is linear topologically nuc iff it is an inverse limit of finite dimensional C^*-algebras; and further, if abelian, isomorphic to the algebra of all scalar sequences with pointwise operations pointwise convergence.

2. Completely positive maps

Let A be a pro-C^*-algebra. Let $M_n(A)$ denote the *algebra of all $n \times n$ matrices A, with the usual algebraic operations and the topology obtained by regarding a direct sum of n^2 copies of A. Since $A = \varprojlim A_p$, as p runs through $S(A)$, we $M_n(A) = \varprojlim_{p \in S(A)} M_n(A_p)$. Thus $M_n(A)$ is a pro-C^*-algebra. Algebraically, $M_n($ $A \otimes M_n(\mathbb{C})$; and since $M_n(\mathbb{C})$ is finite dimensional, all tensor topologies on $M_n(A)$ ag

Lemma 2.1. $b(M_n(A)) = M_n(b(A)) = b(A) \otimes M_n(\mathbb{C})$ *as C^*-algebras.*

Proof. We only have to prove the first equality. If A is a C^*-algebra, $a = [a_{ij}] \in M_n(A)$, it is easily seen that $\max_{i,j} \|a_{ij}\| \leqslant \|a\| \leqslant \Sigma_{i,j} \|a_{ij}\|$. Therefore, inequality holds for any continuous C^*-seminorm p on a pro-C^*-algebra A. Ta supremum over all p yields $\max_{i,j} \sup_p p(a_{ij}) \leqslant \sup_p p(a) \leqslant \Sigma_{i,j} \sup_p p(a_{ij})$. Thus bounded iff each a_{ij} is bounded. The norms on $M_n(b(A))$ and $b(M_n(A))$ must agree by the uniqueness of C^*-norms. This completes the proof.

It is easily seen that for $a = [a_{ij}] \in M_n(A)$, $a \geqslant 0$ iff a is a sum of matrices of the

either A or B is abelian. Following is an analogue of Stinespring's Theorem [27, Chapter IV, Theorem 3.6].

Theorem 2.2. *Let A be a pro-C^*-algebra. Let $B(H)$ denote the C^*-algebra of all bounded linear operators on H.*

(i) If $\pi: A \to B(K)$ is a continuous representation of A, $V: H \to K$ (a Hilbert space) is a bounded linear operator, then $\phi: A \to B(H)$, $\phi(x) = V^*\pi(x)V$ is a continuous completely positive map.
(ii) If $\phi: A \to B(H)$ is a continuous completely positive map, then there exists a Hilbert space K, a continuous representation $\pi: A \to B(K)$, a normal representation $\rho: \phi(A)' \to B(K)$ ($\phi(A)'$ = commutant of $\phi(A)$) and a bounded linear operator $V: H \to K$ such that $\phi(a) = V^*\pi(a)V$, $\rho(x)V = Vx (x \in \phi(A)')$, $\rho(\phi(A)') \subset \pi(A)'$ and $K = [\pi(A)VH]$, closed linear span of $\pi(A)VH$.

(i) is a straightforward verification. For (ii), continuity of ϕ implies that there exists $p \in S(A)$ such that $\|\phi(a)\| \leqslant Mp(a)$ for all $a \in A$, for some $M > 0$. Thus $\phi_p: A_p \to B(H)$, $\phi_p(x_p) = \phi(x)$, $x_p = x + N_p$ is a well defined completely positive map between C^*-algebras to which the C^*-algebras Stinespring's Theorem applies.

COROLLARY 2.3

Let A and B be pro-C^*-algebras.

(i) A unital continuous linear map $\phi: A \to B$ is completely positive (respectively positive) iff $\phi(b(A)) \subset b(B)$ and $\phi(b(A)) \to b(B)$ is a completely positive (respectively positive) map between C^*-algebras. If ϕ is completely positive, then $\phi(a)^*\phi(a) \leqslant \phi(a^*a)$ for all $a \in A$.
*(ii) A is homeomorphically *isomorphic to B iff there exits a unital continuous bijective completely positive map $\phi: A \to B$ such that ϕ^{-1} is continuous and completely positive. In particular, if A and B are σ-C^*-algebras, then A is homeomorphically *isomorphic to B iff A and B are matricially order isomorphic.*

Proof. (i) Let $\phi: A \to B$ be positive (in particular, completely positive). We show that $\phi(b(A)) \subset b(B)$. Let $a \in b(A)$ be positive, and so $0 \leqslant a \leqslant \|a\|_\infty 1$. Hence $0 \leqslant \phi(a) \leqslant \|a\|_\infty \phi(1) = \|a\|_\infty$. Thus $\phi(a) \in b(B)$. Now $\phi(b(A)) \subset b(B)$ follows by writing an arbitrary element as a linear combination of positive ones. Conversely, let given $\phi: A \to B$ be such that $\phi(b(A)) \subset b(B)$ and $\phi: b(A) \to b(B)$ is completely positive. In view of Lemma 3.4, for all n, $\phi_n(b(M_n(A))) = \phi_n(M_n(b(A))) \subset M_n(b(B)) = b(M_n(B))$ and ϕ_n: $b(M_n(A)) \to b(M_n(B))$ are positive linear maps, continuous in the relative topologies from $M_n(A)$ and $M_n(B)$. Now $b(A)$ is dense in A, in fact any $h \geqslant 0$ in A can be approximated by the sequence $h_k = h(1 + (1/k)h^2)^{-1} \geqslant 0$ in $b(A)$. Applying this to each $M_n(A)$ and using $b(M_n(A)) = M_n(b(A))$, each $\phi_n: M_n(A) \to M_n(B)$ is positive.

Let $\phi: A \to B$ be completely positive. For each $q \in S(B)$, $\phi_q = \pi_q \circ \phi: A \to B_q (\pi_q: B \to B_q$ being $\pi_q(y) = (y + N_q))$ is a continuous completely positive map. Identifying B_q with

$\phi_q(a^*a) \leqslant \phi(a^*a)$ for all q in $S(B)$. It follows that $\phi(a)^*\phi(a) \leqslant \phi(a^*a)$ for all $a \in A$. This completes the proof of (i).

For the proof of (ii), we shall need the following. A self adjoint unital linear map $\phi: A \to B$ is a *C^*-homomorphism* if $\phi(h^2) = \phi(h)^2$ for all $h = h^*$ in A.

Lemma 2.4. *Let $\phi: A \to B$ be a continuous bijective C^*-homomorphism between pro-C^*-algebras such that for each $n \geqslant 2$, $\phi_n: M_n(A) \to M_n(B)$ is also a C^*-homomorphism. Then ϕ is a *isomorphism (not necessarily a homeomorphism).*

Proof of lemma. A C^*-homomorphism $\phi: A \to B$ being a positive map, Corollary 2.3 (i) implies that $\phi(b(A)) \subset b(B)$. Applying this to ϕ and ϕ^{-1}, it follows that $\phi(b(A)) = b(B)$ and $\phi: b(A) \to b(B)$ is bijective C^*-isomorphism between C^*-algebras. The same arguments show, in view of Lemma 2.1, that for each n, $\phi_n: b(M_n(A)) = M_n(b(A)) \to M_n(b(B)) = b(M_n(B))$ is also C^*-isomorphism. Thus by [27, Example 1, p. 202], $\phi: b(A) \to b(B)$ is a *isomorphism. Now density of $b(A)$ in A, joint continuity of multiplication in a pro-C^*-algebra and continuity of ϕ implies that $\phi: A \to B$ is a *isomorphism.

Proof of part (*ii*) *of Corollary* 2.3. The inequality in part (i) applied to the completely positive maps ϕ and ϕ^{-1} shows that $\phi(a)^*\phi(a) = \phi(a^*a)$ for all $a \in A$. In particular, ϕ is a C^*-isomorphism. By the same arguments, each ϕ_n is a C^*-isomorphism; and hence ϕ is a *isomorphism by above Lemma. The remaining assertions are trivial.

We note the following consequence of the fact that a positive linear functional on a complete locally m-convex *algebra with 1 maps a bounded set to a bounded set [7].

PROPOSITION 2.5

A positive linear map $\phi: A \to B$ from pro-C^*-algebra A to pro-C^*-algebra B maps a bounded set to a bounded set. If A is a σ-C^*-algebra, then ϕ is continuous.

3. Tensor products

Given pro-C^*-algebras A and B, we are concerned with the following four topologies on the tensor product $A \otimes B$. For a locally convex topology τ on $A \otimes B$, $A \hat{\otimes}_\tau B$ denotes the completion of $(A \otimes B, \tau)$.

(i) [26] projective tensorial topology π: For $p \in S(A)$, $q \in S(B)$, $(p \otimes q)(z) = \inf\{\Sigma_i p(x_i) q(y_i) | z = \Sigma_i x_i \otimes y_i$ in $A \otimes B$ with $x_i \in A$, $y_i \in B\}$; and π is the locally convex topology defined by the seminorms $\{p \otimes q | p \in S(A), q \in S(B)\}$. Each $p \otimes q$ is a submultiplicative seminorm satisfying $(p \otimes q)(z^*) = (p \otimes q)(z)$. Thus $A \hat{\otimes}_\pi B$ is a locally m-convex *algebra, though not a pro-C^*-algebra in general.

(ii) [26] topology ε **of biequicontinuous convergence**: For $p \in S(A)$, $q \in S(B)$, let $U_p^0(1) = \{f \in A^* | |f(x)| \leqslant 1$ for all x such that $p(x) \leqslant 1\}$, $U_q^0(1) = \{g \in B^* | |g(y)| \leqslant 1$ for all y such

and $A\otimes_\varepsilon B$ need not be an algebra.

(iii) [13] projective tensorial pro-C*-topology ν: By a bounded representation (π, H_π) of a *algebra K is meant to be a *homomorphism π of K into the C^*-algebra $B(H_\pi)$. Let $R(K)$ be the collection of all continuous bounded representations of a topological *algebra K. Then $R(K)=\cup\{R_s(K)|s\in S(K)\}$, $R_s(K)=\{\pi\in R(K)|\ \|\pi(x)\|\leqslant s(x)$ for all $x\in K\}$. Now for $p\in S(A)$, $q\in S(B)$, let $R_{p,q}(A\hat{\otimes}_\pi B)=\{\sigma\in R(A\hat{\otimes}_\pi B)|\ \|\sigma(z)\|\leqslant(p\otimes q)(z)$ for all $z\in A\otimes B\}$. Define a C^*-seminorm $\nu_{p,q}(z)=\sup\{\|\sigma(z)\|\ |\sigma\in R_{p,q}(A\hat{\otimes}_\pi B)\}$. The projective tensorial pro-C^*-topology ν is the topology defined by the C^*-seminorms $\{\nu_{p,q}|p\in S(A),\ q\in S(B)\}$.

(iv) [13] injective tensorial pro-C*-topology α: In the above notations let $t_{p,q}(z)=\sup\{\|\pi\otimes\sigma(z)\|\ |\pi\in R_p(A),\ \sigma\in R_q(B)\}$, a C^*-seminorm on $A\otimes B$. The topology α is defined by the C^*-seminorms $\{t_{p,q}|p\in S(A),\ q\in S(B)\}$.

Note that $\varepsilon\leqslant\alpha\leqslant\nu\leqslant\pi$; and for C^*-algebras A and B, the topologies ε, α, ν and π reduce to the topologies respectively due to the injective cross norm $\|\cdot\|_\lambda$, the minimal C^*-norm $\|\cdot\|_{\min}$, the maximal C^*-norm $\|\cdot\|_{\max}$ and the projective cross norm $\|\cdot\|_\gamma$ [27, Chapter IV].

Lemma 3.1. *For pro-C^*-algebras A and B, the following hold*

(i) $A\hat{\otimes}_\nu B=\varprojlim_{p,q} A_p\hat{\otimes}_{\max}B_q$ (ii) $A\hat{\otimes}_\alpha B=\varprojlim_{p,q} A_p\hat{\otimes}_{\min}B_q$

(iii) $A\hat{\otimes}_\pi B=\varprojlim_{p,q} A_p\hat{\otimes}_\gamma B_q$ (iv) $A\hat{\otimes}_\varepsilon B=\varprojlim_{p,q} A_p\hat{\otimes}_\lambda B_q$.

Given pro-C^*-algebras A and B, a Hausdorff topology τ on $A\otimes B$ is an admissible topology [13] if:(i) $(A\otimes B,\tau)$ is a locally m-convex *algebra; i.e., there exists a family $\Gamma=\{r_\alpha|\alpha\in\Delta\}$ of submultiplicative seminorms satisfying $r_\alpha(z^*)=r_\alpha(z)$ for all z such that Γ determines τ. (ii) for each $\alpha\in\Delta$, there exist $p\in S(A)$, $q\in S(B)$ such that $r_\alpha(x\otimes y)\leqslant p(x)q(y)$ for all $x\in A$, $y\in B$; and (iii) given equicontinuous subsets $M\subset A^*$, $N\subset B^*$, $M\otimes N$ is an equicontinuous subset of $(A\hat{\otimes}_\tau B)$. A topology τ on a *algebra K is a pro-C^*-topology if the completion of (K,τ) is a pro-C^*-algebra. It is shown in [13, Proposition 3.1] that if A and B are σ-C^*-algebras and if τ is an admissible pro-C^*-topology on $A\otimes B$ such that $(A\otimes B,\tau)$ is a Q-algebra, then $\tau\leqslant\nu$. The following substantially improves this. It can also be regarded as an analogue of the C^*-algebra result [27, Chapter IV, Theorem 4.19] that if β is a C^*-norm on $A\otimes B$ for C^*-algebras A and B, then β is a cross norm satisfying $\|\cdot\|_{\min}\leqslant\beta(\cdot)\leqslant\|\cdot\|_{\max}$; and if either A or B is abelian, then $\beta(\cdot)=\|\cdot\|_\lambda=\|\cdot\|_{\min}=\|\cdot\|_{\max}$ [27, Chapter IV, Lemma 4.18]. The other half of the following also improves [13, Theorem 3.1] wherein it is shown that for Q-pro-C^*-algebras A and B one being barrelled and the other abelian, $\varepsilon=\alpha$ on $A\otimes B$. In fact, Proposition 3.1 and Theorem 3.1, both of [13] referred to above, reduce to be the usual C^*-algebra results, because, by [23, Proposition 1.14], a Q-pro-C^*-algebra is a C^*-algebra.

Theorem 3.2. *Let A and B be pro-C^*-algebras. Let τ be an admissible pro-C^*-topology on $A\otimes B$. Then $\alpha\leqslant\tau\leqslant\nu$. If either A or B is abelian, then $\varepsilon=\alpha=\tau=\nu$.*

Proof. Let $K_\tau=A\hat{\otimes}_\tau B$. Let (z_α) be a net in $A\otimes B$, $z_\alpha\to 0$ in τ. By [26, Chapter IV, p. 127], the topology of a locally convex space is the topology of uniform convergence

on equicontinuous subsets of its dual. Hence $z_\alpha \to 0$ uniformly on every equicontinu subset of $(A \otimes_\tau B)^*$. In view of admissibility of τ, $z_\alpha \to 0$ uniformly on $M \otimes N$ equicontinuous sets $M \subset A^*$, $N \subset B^*$. Hence $z_\alpha \to 0$ in ε by [26, Chapter III, p. 96],

$$\varepsilon \leqslant \tau.$$

Now given $\gamma \in S(A \hat{\otimes}_\tau B)$, choose $p \in S(A)$, $q \in S(B)$ such that for all x, y, $\gamma(x \otimes$ $p(x)q(y)$. Hence for $z = \Sigma x_i \otimes y_i$ in $A \otimes B$, $\gamma(z) \leqslant \Sigma \gamma(x_i \otimes y_i) \leqslant \Sigma p(x_i) q(y_i)$. By defini of $p \otimes q$, $\gamma(z) \leqslant (p \otimes q)(z)$; and $\tau \leqslant \pi$. Thus

$$\varepsilon \leqslant \tau \leqslant \pi.$$

Again for a net (z_α) in $A \otimes B$, let $z_\alpha \to 0$ in ν, so that for each $p \in S(A)$, $q \in$ $\gamma_{p,q}(z_\alpha) \to 0$. Let $\gamma \in S(A \hat{\otimes}_\tau B)$. Then there exist p and q as in (2) above satisf $\gamma(z) \leqslant (p \otimes q)(z)$ for all z. Since γ is a τ-continuous C^*-seminorm, a faithful representa of the C^*-algebra $(A \hat{\otimes}_\tau B)_\gamma = ((A \hat{\otimes}_\tau B)/\ker \gamma$ with C^*-norm induced by $\gamma)$ as operator algebra gives a τ-continuous bounded representation (σ, H_σ) of A satisfying $\|\sigma(z)\| = \gamma(z) \leqslant p \otimes q(z)$ for all z. Thus $\sigma \in R_{p,q}(A \hat{\otimes}_\pi B)$. Hence $\gamma(z_\alpha) \leqslant \nu_{p,q}(z_\alpha$ showing

$$\tau \leqslant \nu.$$

Finally, we show that $\alpha \leqslant \tau$. For given $p \in S(A)$, $q \in S(B)$, the sets $P_p(A) = \{f \in A^* \mid |f($ $p(x)$ for all x in $A\} \subset A^*$ and $P_q(B) = \{g \in B^* \mid |g(y)| \leqslant q(y)$ for all y in $B\} \subset B^*$ b equicontinuous, $P_p(A) \otimes P_q(B)$ is an equicontinuous subset of $(A \hat{\otimes}_\tau B)^*$. Hence t exist a $\gamma \in S(A \hat{\otimes}_\tau B)$ such that

$$P_p(A) \otimes P_q(B) \subset P_\gamma(A \hat{\otimes}_\tau B) = \{h \in (A \hat{\otimes}_\tau B)^* \mid |h(z)| \leqslant \gamma(z) \text{ for all } z\}.$$

Now for a $\pi \in R_p(A)$, $\xi \in H_\pi$, the linear functional $f_\xi^\pi(x) = \langle \pi(x)\xi, \xi \rangle / \|\xi\|^2$ sati $|f_\xi^\pi(x)| \leqslant p(x)$ for all $x \in A$; and the equicontinuous set M defined as $M = \{f_\xi^\pi \mid \pi \in R$ $\xi \in H_\pi\}$ satisfies $M \subset P_p(A)$. Similarly, $N = \{g_\eta^\sigma \mid \sigma \in R_q(B), \eta \in K_\sigma\} \subset P_q(B)$ where $g_\eta^\sigma($ $\langle \sigma(y)\eta, \eta \rangle / \|\eta\|^2$. It follows from (4) that for all $(\pi, H_\pi) \in R_p(A)$, $(\sigma, K_\sigma) \in R_q(B)$, $\xi \in$ $\eta \in K_\sigma$, $|f_\xi^\pi \otimes g_\eta^\sigma(z)| \leqslant \gamma(z)$ for all $z \in A \hat{\otimes}_\tau B$. Taking $z = \Sigma x_i \otimes y_i$ in $A \otimes B$, this im that

$$\left| \sum_i \frac{\langle (\pi \otimes \sigma)(x_i \otimes y_i)(\xi \otimes \eta), (\xi \otimes \eta) \rangle}{\|\xi \otimes \eta\|^2} \right| = \left| \sum_i \frac{\langle \pi(x_i)\xi, \xi \rangle \langle \sigma(y_i)\eta, \eta \rangle}{\|\xi\|^2 \|\eta\|^2} \right|$$
$$= f_\xi^\pi \otimes g_\eta^\sigma(z) \leqslant \gamma(z).$$

Thus, for all $z \in A \otimes B$, $|\langle (\pi \otimes \sigma)(z)(\xi \otimes \eta), \xi \otimes \eta \rangle| \leqslant \gamma(z) \|\xi \otimes \eta\|^2$. It follows by polarization identity, that for any ξ, ξ' in H_π and η, η' in K_σ, $|\langle \pi \otimes \sigma(z)(\xi \otimes \eta), \xi' \otimes \eta'$ $\gamma(z)|\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle|$. Now taking $\theta = \Sigma_i \xi_i \otimes \eta_i \in H_\pi \otimes K_\sigma$, and taking without lo generality, (η_i) to be an orthonormal set in K_σ, we obtain

$$|\langle (\pi \otimes \sigma)(z)\theta, \theta \rangle| \leqslant \sum_{i,j} |\langle (\pi \otimes \sigma)(z)(\xi_i \otimes \eta_i), \xi_j \otimes \eta_j \rangle|$$

$$\leqslant \gamma(z) \sum_{i,j} |\langle \xi_i \otimes \eta_i, \xi_j \otimes \eta_j \rangle|$$

$$= \gamma(z) \sum \|\xi_i\|^2 = \gamma(z) \|\theta\|^2$$

Since $\pi \otimes \sigma(z)$ is a bounded operator on the completed Hilbert space tensor product $H_\pi \bar{\otimes} K_\sigma$, it follows that (5) holds for all θ in $H_\pi \bar{\otimes} K_\sigma$. Thus, for $\pi \in R_p(A)$, $\sigma \in R_q(B)$, $z \in A \otimes B$ $\theta \in H_\pi \bar{\otimes} K_\sigma$,

$$\begin{aligned} \|\pi \otimes \sigma(z)\theta\|^2 &= \langle (\pi \otimes \sigma)(z^* z)\theta, \theta \rangle \\ &\leqslant \gamma(z^* z)\|\theta\|^2 = \gamma(z)^2 \|\theta\|^2. \end{aligned} \tag{6}$$

Hence for all $z \in A \otimes B$ (and so for all $z \in A \hat{\otimes}_\tau B$), $\|\pi \otimes \sigma(z)\| \leqslant \gamma(z)$, $\pi \otimes \sigma \in R_\gamma(A \hat{\otimes}_\tau B)$. Summarizing, we have shown that given $p \in S(A)$, $q \in S(B)$, there exists a $\gamma \in S(A \hat{\otimes}_\tau B)$ such that $R_p(A) \otimes R_q(B) \subset R_\gamma(A \otimes B)$.

Now if $z_\alpha \to 0$ in τ, it follows from above that for all $p \in S(A)$, $q \in S(B)$, $t_{p,q}(z_\alpha) \to 0$. Hence

$$\alpha \leqslant \tau. \tag{7}$$

Finally, assume that either A or B, say A, is abelian. Then for $p \in S(A)$, $q \in S(B)$, by standard C^*-algebra result [27, p. 212], $\lambda = \|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $A_p \otimes B_q$. Lemma 3.1 implies that $A \hat{\otimes}_\alpha B = A \hat{\otimes}_\varepsilon B = A \hat{\otimes}_\nu B$ with $\alpha = \varepsilon = \nu$; in fact, $\varepsilon_{p,q} = t_{p,q}$. This completes the proof.

Remarks 3.3. (a) A closer examination of the proof reveals that we have, in fact, proved the following stronger assertions.

(i) Any admissible topology τ on $A \otimes B$ satisfies $\varepsilon \leqslant \tau \leqslant \pi$.
(ii) In the derivation of inequality (6), if γ is not a C^*-seminorm but a submultiplicative *seminorm only, then too $\|\pi \otimes \sigma(z)\theta\|^2 \leqslant \gamma(z^* z)\|\theta\|^2 \leqslant \gamma(z^*)\gamma(z)\|\theta\|^2 \leqslant \gamma(z)^2\|\theta\|^2$ holds. Thus, for any admissible (not necessarily pro-C^*) topology τ on $A \otimes B$, $\alpha \leqslant \tau$; and α (respectively π) is the smallest (respectively finest) admissible locally m-convex *algebra topology on $A \otimes B$.
(iii) In general, $\nu < \pi$, since $A \hat{\otimes}_\nu B$ is, under certain circumstances, the enveloping pro-C^*-algebra [11] of the complete locally m-convex *algebra $A \hat{\otimes}_\pi B$ [13].

(b) It is easily seen from [27] that for C^*-algebras A and B, any C^*-topology on $A \otimes B$ is automatically an admissible topology. This suggests: Let τ be any pro-C^*-topology on the tensor product $A \otimes B$ of pro-C^*-algebras A and B. Is τ necessarily an admissible topology?

Our next result shows that the conclusion $\varepsilon = \alpha$ in abelian case of above theorem, in fact, characterizes commutativity of either A or B. It gives a complete pro-C^*-analogue of [27, Chapter IV, Theorem 4.14] and a recent result of Blecher [3]; viz. for C^*-algebras A and B, if $A \hat{\otimes}_\lambda B$ is a Banach algebra with norm λ, then either A or B is abelian. We call the natural ε-calibration $\Gamma = \{\varepsilon_{p,q} | p \in S(A),\ q \in S(B)\}$ an *m^*-calibration* if each $\varepsilon_{p,q}$ satisfies $\varepsilon_{p,q}(uv) \leqslant \varepsilon_{p,q}(u)\varepsilon_{p,q}(v)$, $\varepsilon_{p,q}(u^*) = \varepsilon_{p,q}(u)$ for all u, v in $A \otimes B$. The given pro-C^*-topologies on A and B are denoted by τ_A and τ_B respectively; and $b(A)$ and $b(B)$ carry C^*-topologies unless stated otherwise.

Theorem 3.4. *Let A and B be pro-C^*-algebras. The following are equivalent.*

(1) *Either A or B is abelian.*
(2) *Any continuous pure state ω on $A \hat{\otimes}_\alpha B$ is of form $\omega = \omega_1 \otimes \omega_2$, ω_1 and ω_2 being continuous pure states on A and B respectively.*

(3) $A\hat{\otimes}_{\varepsilon}B = A\hat{\otimes}_{\alpha}B$ *with* $\varepsilon = \alpha$ *and the natural* ε*-calibration on* $A\hat{\otimes}_{\varepsilon}B$ *is an* m^**-calibration.*
(3′) *The natural* ε*-calibration on* $A\hat{\otimes}_{\varepsilon}B$ *is an* m^**-calibration.*
(4) $b(A)\hat{\otimes}_{\lambda}b(B) = b(A)\hat{\otimes}_{\min}b(B)$ *and* $\lambda = \|\cdot\|_{\min}$.
(4′) $b(A)\hat{\otimes}_{\lambda}b(B)$ *is a Banach algebra.*
(5) $A\hat{\otimes}_{\varepsilon}b(B) = A\hat{\otimes}_{\alpha}b(B)$ *with* $\varepsilon = \alpha$ *and the natural* ε*-calibration on* $A\hat{\otimes}_{\varepsilon}b(B)$ *is an* m^**-calibration.*
(5′) *The natural* ε*-calibration on* $A\hat{\otimes}\,b(B)$ *is an* m^**-calibration.*
(6) $b(A)\hat{\otimes}_{\varepsilon}B = b(A)\hat{\otimes}_{\alpha}B$ *with* $\varepsilon = \alpha$ *and the natural* ε*-calibration on* $b(A)\hat{\otimes}_{\varepsilon}B$ *is an* m^**-calibration.*
(6′) *The natural* ε*-calibration on* $b(A)\hat{\otimes}_{\varepsilon}B$ *is an* m^**-calibration.*

Lemma 3.5. *Let* R *be a pro-*C^**-algebra. Let* E *be *subalgebra of* R *containing the identity of* R *such that* E *is a Banach *algebra with some norm* $|\cdot|$. *Then* $E \subset b(A)$ *and on* E, $\|\cdot\|_{\infty} \leqslant |\cdot|$.

Proof. By continuity of involution in $(R, |\cdot|)$, we assume $|x^*| = |x|$ for all $x \in E$. Let $B = \{x \in E \,|\, |x| \leqslant 1\}$. Then B is an absolutely convex *idempotent containing the identity 1 of R. By standard Banach *algebra arguments, every positive linear functional f on R, restricted to E, is $|\cdot|$-continuous satisfying $|f(x)| \leqslant f(1)|x|$. Since the dual R^* of R is a complex linear span of continuous positive functionals, B turns out to be $\sigma(R, R^*)$ bounded, hence bounded in the topology of R. Now, in R, it is easy to verify that $K = \{x \in R \,|\, \|x\|_{\infty} \leqslant 1\}$ is the largest (under inclusion) bounded absolutely convex *idempotent. Thus $B \subset K$, $E \subset b(R)$ and $\|\cdot\|_{\infty} \leqslant |\cdot|$ on E.

Proof of theorem. (1)$\Rightarrow$(2). First of all, for any A and B, not necessarily abelian, let $K(A\hat{\otimes}_{\alpha}B)$ be the set of all continuous states on $A\hat{\otimes}_{\alpha}B$. For each $j = (p, q) \in S(A) \times S(B)$, Let $U_j = \{z \in A\hat{\otimes}_{\alpha}B \,|\, t_{p,q}(z) \leqslant 1\}$, R_j be the C^*-algebra $A\hat{\otimes}_{\alpha}B/\ker t_{p,q}$ with the C^*-norm induced by $t_{p,q}$. Then R_j is *isomorphic to the C^*-algebra $A_p\hat{\otimes}_{\min}B_q$. Let $K_j(A\hat{\otimes}_{\alpha}B) = \{f \in K(A\hat{\otimes}_{\alpha}B) \,|\, f$ is bounded on $U_j\}$. Then from [4], the following hold.

(a) $K(A\hat{\otimes}_{\alpha}B) = U_j K_j(A\hat{\otimes}_{\alpha}B)$.
(b) For the sets of extreme points, $E(K(A\hat{\otimes}_{\alpha}B)) = U_j E(K_j(A\hat{\otimes}_{\alpha}B))$.
(c) $K_j(A\hat{\otimes}_{\alpha}B)$ is in bijective correspondence with $K(R_j)$ under the map $f \in K_j(A\hat{\otimes}_{\alpha}B) \to f_j$, $f_j(z_j) = f(z)$ where for $z \in A \otimes B$, $z_j \in R_j$ is $z_j = z + \ker t_{p,q}$; and this correspondence preserves the weak* topologies.

Now assume (1) say B is abelian. For all q in $S(B)$, the C^*-algebras B_q are abelian. Let ω be a continuous pure state on $A\hat{\otimes}_{\alpha}B$, so that, by [4, Corollary 4.3], ω is an extreme point of $K(A\hat{\otimes}_{\alpha}B)$. By above, there exists a $j = (p, q) \in S(A) \times S(B)$ such that ω_j is a pure state on $R_j = A_p\hat{\otimes}_{\min}B_q$. The C^*-algebra result [27, p. 211] gives pure states ω'_1 of A_p and ω'_2 of B_q satisfying $\omega_j = \omega'_1 \otimes \omega'_2$. Again from above and [4, Theorem 4.3], $\omega_1(x) = \omega'_1(x_p)$, $\omega_2(y) = \omega'_2(y_q)$ define continuous pure states on A and B satisfying $\omega = \omega_1 \otimes \omega_2$.
(2)$\Rightarrow$(1). This can be shown exactly as in [27, Chapter IV, Theorem 4.14] by using the facts [11] that a pro-C^*-algebra R admits sufficiently many bounded continuous

in $S(A)\times S(B)$, $A_p\hat{\otimes}_\lambda B_q$ becomes a Banach algebra with the injective cross-norm λ. Hence by [3, Corollary 4, p. 123] either A_p or B_q is abelian; and so C^*-algebra arguments [27, Theorem 4.14, p. 211] gives $A_p\hat{\otimes}_\lambda B_q = A_p\hat{\otimes}_{\min} B_q$ with $\lambda = \|\cdot\|_{\min}$. This gives (3). Similarly it follows that $(4')\Rightarrow(4)$, $(5')\Rightarrow(5)$ and $(6')\Rightarrow(6)$.

$(3)\Rightarrow(1)$. Observe the following.

(i) $b(A\hat{\otimes}_\alpha B)$ is a C^*-algebra with C^*-norm $\|z\|_{\infty,\alpha} = \sup\{t_{p,q}(z)|p\in S(A),\ q\in S(B)\}$.

(ii) Since the ε-calibration is an m^*-calibration, $A\hat{\otimes}_\varepsilon B$ is a complete locally m-convex *algebra; and its bounded part, defined as $b(A\hat{\otimes}_\varepsilon B) = \{z\in A\hat{\otimes}_\varepsilon B|\sup_{p,q}\varepsilon_{p,q}(z)<\infty\}$ is a Banach *algebra with norm $\|z\|_{\infty,\varepsilon} = \sup\{\varepsilon_{p,q}(z)|p\in S(A), q\in S(B)\} = \|z^*\|_{\infty,\varepsilon}$. Now assume $A\hat{\otimes}_\alpha B = A\hat{\otimes}_\varepsilon B$. We assert that $b(A\hat{\otimes}_\alpha B) = b(A\hat{\otimes}_\varepsilon B)$. Indeed, for all p,q in $S(A)\times S(B)$, $\varepsilon_{p,q}\leqslant t_{p,q}$ and so $\|\cdot\|_{\infty,\varepsilon}\leqslant\|\cdot\|_{\infty,\alpha}$ with the result $b(A\hat{\otimes}_\alpha B)\subset b(A\hat{\otimes}_\varepsilon B)$. Further, above (ii) in the light of Lemma 3.5, applied to the pro-C^*-algebra $A\hat{\otimes}_\alpha B$, implies that $b(A\hat{\otimes}_\varepsilon B)\subset b(A\hat{\otimes}_\alpha B)$ and $\|\cdot\|_{\infty,\varepsilon} = \|\cdot\|_{\infty,\alpha}$.

Further, $b(A)\otimes b(B)\subset b(A\hat{\otimes}_\alpha B)$ and the norm $\|\cdot\|_{\min}$ on $b(A)\otimes b(B)$ is [27, p. 207] $\|z\|_{\min} = \sup\{\|\pi\otimes\sigma(z)\|\,|\pi\in R(b(A)), \sigma\in R(b(B))\}\geqslant\|z\|_{\infty,\alpha}$ by definition of $\|\cdot\|_{\infty,\alpha}$. But $\|\cdot\|_{\min}$ is the smallest among all C^*-norms on $b(A)\otimes b(B)$ [27, Proposition 4.19, p. 216]. Hence $\|\cdot\|_{\min} = \|\cdot\|_{\infty,\alpha} = \|\cdot\|_{\infty,\varepsilon}$ on $b(A)\otimes b(B)$ and so on $b(A)\hat{\otimes}_{\min} b(B)$. Finally, the λ-norm on $b(A)\otimes b(B)$ is [27, p. 188], with $z = \Sigma x_i\otimes y_i$, $\lambda(z) = \sup\{|\Sigma f(x_i)g(y_i)|\ |f\in b(A)^*, |f(X)|\leqslant\|x\|_\infty, g\in b(B)^*, |g(y)|\leqslant\|y\|_\infty\}\leqslant\|z\|_{\infty,\varepsilon}$ by definition of $\|\cdot\|_{\infty,\varepsilon}$.

Thus, on $b(A)\otimes b(B)$, $\|\cdot\|_{\infty,\varepsilon}\leqslant\lambda(\cdot)\leqslant\|\cdot\|_{\min}\leqslant\|\cdot\|_{\infty,\alpha} = \|\cdot\|_{\infty,\varepsilon}$ with the result that $\lambda(\cdot) = \|\cdot\|_{\min}$. It follows [27, Theorem 4.14, p. 211] that either $b(A)$ or $b(B)$ is abelian. But $b(A)$ is dense in A, $b(B)$ is dense in B and the multiplication in a pro-C^*-algebra is jointly continuous. It follows that either A or B is abelian. $(1)\Leftrightarrow(4)$ is a consequence of C^*-algebra theory together with density arguments as above. Similarly one gets $(1)\Leftrightarrow(5)$ and $(1)\Leftrightarrow(6)$ from $(1)\Leftrightarrow(3)$. This completes the proof.

Remark 3.6. In the above theorem, to conclude $(3)\Rightarrow(1)$ and similar other implications, the hypothesis that the ε-calibration is an m^*-calibration cannot be omitted, even for C^*-algebras. Take A to be a non-abelian finite dimensional C^*-algebra. Then for all C^*-algebras B, $A\hat{\otimes}_\varepsilon B = A\hat{\otimes}_\pi B$ and $\varepsilon=\pi$, i.e., λ and γ, and so λ and $\|\cdot\|_{\min}$ are equivalent. On the other hand $\lambda = \|\cdot\|_{\min}$ forces A or B to be abelian.

PROPOSITION 3.7

Let $\phi_1: A_1\to B_1$, $\phi_2: A_2\to B_2$ be continuous completely positive maps between pro-C^*-algebras. Then

$$\phi_1\otimes\phi_2: A_1\otimes A_2\to B_2,\quad (\phi_1\otimes\phi_2)(x_1\otimes x_2) = \phi_1(x_1)\otimes\phi_2(x_2)$$

is an $\alpha-\alpha$ continuous (respectively $v-v$ continuous) completely positive map that extends as a completely positive map $\phi_1\otimes\phi_2: A_1\hat{\otimes}_\alpha A_2\to B_1\hat{\otimes}_\alpha B_2$ (respectively, $\phi_1\otimes\phi_2: A_1\hat{\otimes}_v A_2\to B_1\hat{\otimes}_v B_2$).

Proof. We shall prove for the topology α. The other assertion can be similarly

of quotient C^*-algebras $(A_i)_{p_i} \to (B_i)_{q_i}$. By the corresponding property of C^*-algebras [27, p. 218], there is a completely positive map $\theta:(A_1)_{p_1}\hat{\otimes}_{\min}(A_2)_{p_2}\to(B_1)_{q_1}\hat{\otimes}_{\min}(B_2)_{q_2}$. The desired map $\psi_{p,q}$ is the composite map $A_1\hat{\otimes}_\alpha A_2\to(A_1)_{p_1}\hat{\otimes}_{\min}(A_2)_{p_2}\to(B_1)_{q_1}\hat{\otimes}_{\min}(B_2)_{q_2}$. This completes the proof.

PROPOSITION 3.8

Let A and B be pro-C^*-algebras. Let B^0 (respectively B^*) denote the algebraic dual (respectively topological dual) of B. For a linear functional ω on $A\otimes B$, let $T_\omega:A\to B^0$ be defined by $\langle y, T_\omega x\rangle=\langle x\otimes y,\omega\rangle$.

(1) ω is a state on $A\otimes B$ iff T_ω is a complete state map (i.e., T_ω is a completely positive map such that $T_\omega(1)$ is a state on B).
(2) Further, ω is continuous in the topology v (so that $\omega\in(A\hat{\otimes}_v B)^*$) iff $T_\omega(A)\subset B^*$ and $T_\omega:A\to B^*$ is continuous, where B^* carries the topology τ_b of uniform convergence on all bounded subsets of B.

Proof. (1) is exactly as in [27, Chapter IV, Proposition 4·6]. Further, let ω be v-continuous. There exist $(p,q)\in S(A)\times S(B)$ and a scalar $k>0$ such that for all $z\in A\otimes B$, $|\omega(z)|\leqslant kv_{p,q}(z)$. Hence for all $x\in A$, $y\in B$, $|\langle y, T_\omega(x)\rangle|=|\langle x\otimes y,\omega\rangle|\leqslant kv_{p,q}(x\otimes y)\leqslant kp\otimes q(x\otimes y)\leqslant kp(x)q(y)$. Hence for $K\subset B$ bounded with $q(y)\leqslant M_{q,k}(y\in K)$, $|\langle y, T_\omega x\rangle|\leqslant kM_{q,k}p(x)(x\in A)$ showing $T_\omega A\subset B^*$ and T_ω is continuous in τ_b. Conversely, let $T_\omega:A\to B^*$ be τ_b continuous. By general theory of topological tensor products [26], $\omega\in(A\hat{\otimes}_\pi B)^*$. By the GNS construction [11, Theorem 3.4] on the complete locally m-convex *algebra $A\hat{\otimes}_\pi B$, there exists a continuous bounded representation $\pi_\omega:A\hat{\otimes}_\pi B\to B(H_\pi)$ on a Hilbert space H_π having a cyclic vector ξ_0 such that $\omega(z)=\langle\pi_\omega(z)\xi_0,\xi_0\rangle$ and $\|\pi_\omega(z)\|\leqslant(p\otimes q)(z)(z\in A\otimes B)$ for some $(p,q)\in S(A)\times S(B)$. Thus $\pi_\omega\in R_{p,q}(A\hat{\otimes}_\pi B)$; and for all z, $|\langle z,\omega\rangle|=|\langle\pi_\omega(z)\xi_0,\xi_0\rangle|\leqslant\|\pi_\omega(z)\|\,\|\xi_0\|^2\leqslant v_{p,q}(z)\,\|\xi_0\|^2$ showing that ω is a continuous in the topology v.

4. Nuclear pro-C^*-algebras

Following a suggestion in [23], a pro-C^*-algebra is called *nuclear* if for each $p\in S(A)$, the C^*-algebra $(A_p,\|\cdot\|_p)$ is a nuclear C^*-algebra [16], [17] in the sense that for any C^*-algebra B, $\|\cdot\|_{\min}=\|\cdot\|_{\max}$ on $A_p\otimes B$. Thus a nuclear C^*-algebra is a nuclear pro-C^*-algebra; and a nuclear pro-C^*-algebra is an inverse limit of nuclear C^*-algebras. Commutative pro-C^*-algebras, the matrix algebra $M_n(A)$ over a nuclear pro-C^*-algebra A and a pro-C^*-algebra of type I (in the sense that all continuous bounded representations are of type I) are all nuclear pro-C^*-algebras.

PROPOSITION 4.1

Let a pro-C^*-algebra $A=\varprojlim_{\alpha\in\Delta}B_\alpha$, an inverse limit of C^*-algebras with the maps $\pi_\alpha:A\to B_\alpha$ of the inverse system assumed surjective. Then A is a nuclear pro-C^*-algebra iff each B_α is a nuclear C^*-algebra.

Proof. One way is obvious, since $A=\varprojlim_{p\in S(A)}A_p$. Conversely, let $A=\varprojlim_{\alpha\in\Delta}B_\alpha$ where each B_α is a nuclear C^*-algebra. Then the family $\{p_\alpha|\alpha\in\Delta\}$ of continuous C^*-seminorms determines the topology of A, where $p_\alpha(x)=\|\pi_\alpha(x)\|_\alpha$, π_α being the

*homomorphism from A to the C^*-algebra $(B_\alpha, \|\cdot\|_\alpha)$. The B_α is *isomorphic to the C^*-algebra $A_\alpha = A/\ker p_\alpha$ with C^*-norm $\|x + \ker p_\alpha\|_{p_\alpha} = p_\alpha(x)$ [1, Theorem 2.4]. Thus A_α is nuclear. Given $p \in S(A)$, by continuity, there exists an $\alpha \in \Delta$ such that $p \leqslant p_\alpha$. Thus $\phi: A_\alpha \to A_p$, $\phi(x + \ker p_\alpha) = x + \ker p$ is a well defined continuous surjective *homomorphism, and A_p is *isomorphic to the quotient C^*-algebra $A_\alpha/\ker \phi$, which is nuclear, since A_α is nuclear [6], [17]. This completes the proof.

For pro-C^*-algebras A and B, the identity map $A \otimes B \to A \otimes B$ extends to a continuous surjective *homomorphism $\psi: A \hat{\otimes}_v B \to A \hat{\otimes}_\alpha B$. The following shows that A is nuclear iff ψ is a homeomorphism for all B.

Theorem 4.2. *For a pro-C^*-algebra A, the following are equivalent.*

(1) *A is nuclear.*
(2) *For all pro-C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$ with $\alpha = v$.*
(3) *For all C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$ with $\alpha = v$.*
(4) *For all pro-C^*-algebras B (respectively C^*-algebras B), there is only one admissible pro-C^*-topology on $A \otimes B$.*

Further, if A is a σ-C^*-algebra, then above are equivalent to any of the following.

(5) *For all σ-C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$.*
(6) *For all C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$.*
(7) *For all σ-C^*-algebras B (respectively C^*-algebras B), the topology v on $A \otimes B$ is faithful.*

An admissible topology τ on $A \otimes B$ is faithful [12] *if the map $i_\tau: A \hat{\otimes}_\tau B \to A \hat{\otimes}_\varepsilon B \subset B(A^*, B^*)$, $i_\tau(z) = (x' \otimes y')(z)$, $x' \in A^*, y' \in B^*$ is one–one. The following improves* [12, *Proposition* 3.3].

Lemma 4.3. *The injective tensorial pro-C^*-topology α on $A \otimes B$ is faithful.*

Proof. The map $i_\alpha: A \hat{\otimes}_\alpha B \to A \hat{\otimes}_\varepsilon B$ is the unique continuous linear extension of the identity map $i: A \otimes B \to A \otimes B$, $i(z) = z$. Let $z \in A \hat{\otimes}_\alpha B$ with $i_\alpha(z) = 0$. Let (z_λ) be a net in $A \otimes B$, $z_\lambda = \Sigma_{i=1}^{k_\lambda} x_i^{(\lambda)} \otimes y_i^{(\lambda)} \to_\alpha z$. Then for all $f \in A^*$, $g \in B^*$, $\lim_\lambda i_\alpha(z_\lambda)(f, g) = \lim_\lambda f \otimes g(z_\lambda) = f \otimes g(z) = 0$. To show $z = 0$, we show that $t_{p,q}(z) = 0$ for all $(p, q) \in S(A) \times S(B)$. Let $\pi \in R_p(A)$, $\sigma \in R_q(B)$, $\xi_1 \in H_\pi$, $\eta_1 \in H_\pi$, $\xi_2 \in H_\sigma$, $\eta_2 \in H_\sigma$. Define f and g by $f(x) = \langle \pi(x)\xi_1, \eta_1 \rangle$, $g(y) = \langle \sigma(y)\xi_2, \eta_2 \rangle$. Then

$$\begin{aligned} &\langle \pi \otimes \sigma(z)(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2 \rangle \\ &= \lim_\lambda \left(\left\langle \sum_1^{k_\lambda} \pi(x_i^{(\lambda)}) \otimes \sigma(y_i^{(\lambda)}) \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \right\rangle \right) \\ &= \lim_\lambda \sum f(x_i^{(\lambda)}) g(y_i^{(\lambda)}) = \lim_\lambda f \otimes g(z_\lambda) = 0. \end{aligned}$$

Hence $\pi \otimes \sigma(z) = 0$ on the completed Hilbert space tensor product $H_\pi \bar{\otimes} H_\sigma$. It follows that $t_{p,q}(z) = 0$.

Lemma 4.4. *The following are equivalent for pro-C^*-algebras A and B.*

(a) *$A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$ with $\alpha = v$.*

(c) *For all* $(p,q)\in S(A)\times S(B)$, $t_{p,q}(\cdot)=v_{p,q}(\cdot)$ on $A\otimes B$.

Proof. (c)$\Leftrightarrow$(b)$\Rightarrow$(a) is a consequence of uniqueness of C^*-norm and Lemma 3.1. We show (a)$\Rightarrow$(b). Let $p\in S(A)$, $q\in S(B)$ and $\pi_p:A\to A_p$, $\pi_q:B\to B_q$ be the quotient maps. Then $\pi_p\hat{\otimes}\pi_q:A\hat{\otimes}_\alpha B\to A_p\hat{\otimes}_{\min}B_q$ defines a continuous *homomorphism. By [27, Proposition 4.7, p. 207], the continuous *homomorphisms $\phi:A\to A_p\hat{\otimes}_{\min}B_q=C$ (say), $\psi:B\to C$, $\phi(x)=x_p\otimes 1_q$, $\psi(y)=1_p\otimes y_q$ give continuous *homomorphism $\eta:A\hat{\otimes}_v B\to C$ satisfying $\eta(z)=(\pi_p\otimes\pi_q)(z)$ on $A\otimes B$. By the assumption (a), $\eta=\pi_p\hat{\otimes}\pi_q$; and π_p and π_q being surjective, $A_p\hat{\otimes}_{\min}B_q=(\pi_p\hat{\otimes}\pi_q)(A\hat{\otimes}_\alpha B)=\eta(A\hat{\otimes}_v B)\subset A_p\hat{\otimes}_{\max}B_q$. The continuous *homomorphism $k:A_p\otimes_{\min}B_q\to A_p\hat{\otimes}_{\max}B_q$ so defined is the extension of the identity map. It follows that for all $z\in A_p\otimes B_q$, $\|z\|_{\max}\leqslant\|z\|_{\min}\leqslant\|z\|_{\max}$, and (b) follows.

Proof of theorem 4.2. Lemma 3.1 gives (1)$\Rightarrow$(2)$\Rightarrow$(3). (2)$\Leftrightarrow$(4) is a consequence of Theorem 3.2. (3)$\Rightarrow$(1) follows from (2)$\Rightarrow$(1), which is a consequence of Lemma 4.4. The remaining assertions on σ-C^*-algebras follow from the open mapping theorem and Lemma 4.3.

It is shown in [23] that the functor $b(\cdot):A\to b(A)$ from pro-C^*-algebras to C^*-algebras is not well behaved with respect to tensor products. Still, the following main result of this section holds giving an aesthetically pleasing nuclearity criterion.

Theorem 4.5. *A pro-C^*-algebras A is nuclear iff $b(A)$ is a nuclear C^*-algebra.*

Lemma 4.6. *Let A be a pro-C^*-algebra. Let $p\in S(A)$, $I_p=b(A)\cap\ker p$. The quotient C^*-algebra $b(A)/I_p$ with the quotient C^*-norm $\|\cdot\|_{\infty,p}$ induced by the C^*-norm on $b(A)$ is isometrically *isomorphic to the C^*-algebra A_p.*

Proof. Let $\pi_p:A\to A_p$ be $\pi_p(x)=x+\ker p$ so that $\pi_p(A)=A_p$. Since $b(A)$ is dense in A, [23, Proposition 1.11] implies that $\pi_p(b(A))=A_p$. The *ideal I_p in $b(A)$ is closed in $\|\cdot\|_\infty$ due to continuity of embedding $b(A)\to A$. Since $I_p=\ker\psi_p$ where $\psi_p=\pi_{p|b(A)}$, $b(A)/I_p$ is *isomorphic to A_p.

Proof of theorem 4.5. Nuclearity of the C^*-algebra $b(A)$ implies [6, Corollary 4] that $(I_p,\|\cdot\|_\infty)$ and $(b(A)/I_p,\|\cdot\|_{\infty,p})$ are nuclear C^*-algebras for each $p\in S(A)$. By Lemma 4.6, A_p is nuclear for all p, and hence A is nuclear. Conversely, let A be a nuclear pro-C^*-algebra. For each $p\in S(A)$, the C^*-algebra A_p can be regarded as a C^*-algebra of operators on some Hilbert space H_p by taking a faithful representation σ_p. Then $\theta_p=\sigma_p\circ\psi_p$ gives a representation of $b(A)$ on H_p such that $\theta_p(b(A))=\sigma_p(A_p)$ is a nuclear C^*-algebra. By [6], [17], the von Neumann algebra $[\theta_p(b(A))]^{cc}$ (second commutant, identified with the bidual $[\theta_p(b(A)]^{**}$) is semidiscrete. By [27, Chapter IV, Lemma 2.2], θ_p extends as a surjective normal homomorphism $\tilde{\theta}_p:b(A)^{**}\to[\theta_p(b(A))]^{cc}$. Thus $\{\tilde{\theta}_p|p\in S(A)\}$ is a faithful family of normal representations of the von Neumann algebra $b(A)^{**}$ such that $\tilde{\theta}_p(b(A)^{**})=\theta_p(b(A))^{cc}$ is semidiscrete. Hence by [10, Corollary 3.3], $b(A)^{**}$ is semidiscrete, and so $b(A)$ is a nuclear C^*-algebra [10, Theorem 6.4].

Remark 4.7. Lemma 4.6 depends only on the fact that $b(A)$ is dense in A and is continuously embedded in A. Thus, in above theorem, we have a slightly stronger

conclusion. Let A be a pro-C^*-algebra. Let B be a *subalgebra of A such that B is dense in A and B is a C^*-algebra with some norm $\|\cdot\|$ so that $(B, \|\cdot\|) \to A$ is a continuous embedding. If B is a nuclear C^*-algebra, then A is a nuclear pro-C^*-algebra. This we shall need elsewhere.

COROLLARY 4.8

(1) *Let a pro-C^*-algebra $A = \Pi_\alpha A_\alpha$, a product of pro-C^*-algebras. Then A is nuclear iff each A_α is nuclear. In particular, arbitrary products of nuclear C^*-algebras are nuclear pro-C^*-algebras.*
(2) *Let I be a closed ideal of a pro-C^*-algebra A such that A/I is complete (in particular, let A be a σ-C^*-algebra). Then A is nuclear iff both I and A/I are nuclear.*
(3) *Let A and B be σ-C^*-algebras. Let $\phi: A \to B$ be a surjective *homomorphism. Then A is nuclear iff $\ker\phi$ and B are nuclear.*
(4) *Let p be a continuous C^*-seminorm on a σ-C^*-algebra A. Then A is nuclear iff $\ker p$ and the C^*-algebra A_p are nuclear.*
(5) *A closed *subalgebra B of a nuclear pro-C^*-algebra A is nuclear if B satisfies either of the following:*

(i) *B is a hereditary subalgebra of A.*
(ii) *There exists a continuous unital completely positive projection from A onto B.*

(6) *Let A be a σ-C^*-algebra. Suppose there exists an increasing sequence (B_n) of σ-C^*-algebras such that*

(i) *each B_n is a closed *subalgebra of A containing the identity of A,*
(ii) *each B_n is nuclear,*
(iii) *$\cup_n B_n$ is dense in A.*

Then A is nuclear.

Proof. (1) For $A = \Pi_\alpha A_\alpha$, $b(A) = \oplus_\alpha b(A_\alpha) = \{x = (x_\alpha) \in A \mid x_\alpha \in b(A_\alpha)$ for all α and $\sup \|x_\alpha\|_{\infty,\alpha} < \infty\}$, $\|\cdot\|_{\infty,\alpha}$ being the C^*-norm on $b(A_\alpha)$ and $\|x\|_\infty = \sup_\alpha \|x_\alpha\|_{\infty,\alpha}$ being the norm on $b(A)$. Then $b(A)^{**} = \oplus_\alpha b(A_\alpha)^{**}$. By [10, Proposition 3.1], $b(A)^{**}$ is semidiscrete iff each $b(A_\alpha)$ is nuclear. The assertion follows from Theorem 4.5.
(2) For each p, $J_p = I/(\ker p \cap I)$ is a closed ideal of the C^*-algebra A_p; and $I = \varprojlim J_p$, $A/I = \varprojlim A_p/J_p$. Hence A is nuclear iff A_p is nuclear iff for each p, J_p and A_p/J_p are nuclear C^*-algebras [6] iff I and A/I are nuclear pro-C^*-algebras by Proposition 4.1.
(3) ϕ is continuous, and the quotient σ-C^*-algebra $A/\ker\phi$ is homeomorphically *isomorphic to B giving $A = \ker\phi \oplus B$, a topological direct sum. The conclusion follows from (1).
(4) By the open mapping theorem, the quotient topology on the σ-C^*-algebra $A/\ker p = A_p$ coincides with the C^*-algebra topology on A_p.
(5) (i) If B is hereditary, then $b(B)$ is a $\|\cdot\|_\infty$-closed hereditary C^*-subalgebra of $b(A)$. The conclusion follows from Theorem 4.5, as a hereditary subalgebra of nuclear C^*-algebra is nuclear [17]. (ii) follows similarly using Corollary 2.3 (i), Theorem 4.5 and the fact that a subalgebra of a nuclear C^*-algebra K is nuclear if there exists a completely positive projection of norm 1 from K onto the subalgebra.
(6) We shall use the following from [2] which is a modification of [1, Lemma 2.1].

Lemma 4.9 *Let A be a pro-C^*-algebra, Let $x \in A$. For each $n = 1, 2, \ldots$, Let $x_n = x(1 + (1/n)x^*x)^{-1}$. Then each $x_k \in b(A)$ and $x_n \to x$ in A.*

We claim that $\cup_n b(B_n)$ is dense in A. Take $x \in A$. Choose a sequence (x_k) in $\cup_n B_n$, say $x_k \in B_{n_k}$ so that $x_k \to x$. We can assume (n_k) to be non-decreasing. By above lemma, for each k and each $n = 1, 2, \ldots$, $x_{k,n} = x_k(1 + (1/n)x_k^* x_k)^{-1} \in b(B_{n_k})$, $x_k = \lim_{n\to\infty} x_{k,n}$ in B_{n_k}. Then $x = \lim_k x_k = \lim_k \lim_n x_k(1 + (1/n)x_k^* x_k)^{-1}$ in A and $x_k(1 + (1/n)x_k^* x_k)^{-1} \in \cup_n b(B_n)$ for all k, n. Hence $\cup_n b(B_n)$ is dense in A. Now for each n, $b(B_n) \subset b(A)$, $\cup_n b(B_n) \subset b(A)$. Let K be the closure of $\cup_n b(B_n)$ in the C^*-algebra $b(A)$. Then the C^*-algebra K is continuously embedded in A with dense range. By Theorem 4.5, each $b(B_n)$ is a nuclear C^*-algebra, with the result, [15, Proposition 11.3.12, p. 859] implies that K is nuclear C^*-algebra. The conclusion follows from Remark 4.7.

Remark 4.10. We could not establish a more general result involving arbitrary pro-C^*-algebras with arbitrary inductive limit. However, the following particular case can be similarly established.

PROPOSITION

Let A be a pro-C^*-algebra. Suppose there exists a family $\{B_\alpha | \alpha \in \Delta\}$ of C^*-algebras such that

(i) *each B_α is a closed *subalgebra of A containing the identity of A,*
(ii) *each B_α is a nuclear C^*-algebra,*
(iii) *given α, β in Δ, there exists a $\gamma \in \Delta$ such that $B_\alpha \cup B_\beta \subset B_\gamma$,*
(iv) *$\cup_\alpha B_\alpha$ is dense in A.*

Then A is a nuclear pro-C^*-algebra.

Finally, we aim to discuss the analog of a basic result of Lance [16, Theorem 3.4] using Proposition 3.8.

Theorem 4.11. *Let A be a pro-C^*-algebra. If A is nuclear, then for any pro-C^*-algebra B, every continuous complete state map from A to the strong dual B^* can be approximated in simple weak* convergence by continuous complete state maps from A to B^* of finite rank.*

Lemma 4.12. *Let A and B be pro-C^*-algebras. Let f be a continuous state on $A \hat{\otimes}_\alpha B$. Then the complete state map $T_f: A \to B^*$, $\langle y, T_f x \rangle = \langle x \otimes y, f \rangle$ can be approximated in simple weak* convergence by complete state maps of finite rank.*

Proof. Let $(p, q) \in S(A) \times S(B)$. Letting the C^*-algebras A_p and B_q act faithfully on Hilbert spaces H_p and K_q respectively, $A_p \otimes_{\min} B_q$ acts faithfully on $H_p \bar{\otimes} K_q$; and by C^*-theory, the state space $K(A_p \hat{\otimes}_{\min} B_q)$ is the weak* closed convex hull of $D_j(0)$, $j = (p, q)$, where $D_j(0) = \{\omega_\xi | \xi = \Sigma \xi_i \otimes \eta_i$ unit vectors in $H_p \otimes K_q\}$ and $\omega_\xi(z) = \langle z_\xi, \xi \rangle$. Let $\pi_j: A \hat{\otimes}_\alpha B \to A \hat{\otimes} B / \ker t_{p,q} = A_p \hat{\otimes}_{\min} B_q$ be the quotient map. Let $D_j = \{\omega_\xi \circ \pi_j | \omega_\xi \in D_j(0)\}$, $D = \cup_j \{D_j | j \in S(A) \times S(B)\}$. Then in the light of statement (a), (b), (c) based on [4] in the proof of (1)$\Rightarrow$(2) of Theorem 3.4, $K_j(A \hat{\otimes}_\alpha B)$ is the weak* closed convex hull of D_j; and $\overline{co} D = \overline{co} \cup_j \{D_j | j \in S(A) \times S(B)\} = \overline{co} \cup \{\overline{co} D_j | j \in S(A) \times S(B)\} = \overline{co} \cup_j \{K_j(A \hat{\otimes}_\alpha B)\} = \overline{co} K(A \hat{\otimes}_\alpha B) = K(A \hat{\otimes}_\alpha B)$, being weak* closed and convex [6].

The lemma follows from this, since for $f \in D$, T_f is a continuous complete state map of finite rank as in Proposition 3.8.

Proof of theorem 4.11. Let $T: A \to B^*$ be a continuous complete state map. By Proposition 3.8, there is an $f \in K(A \hat{\otimes}_\nu B)$ such that $T = T_f$. By Theorem 4.2, $\alpha = \nu$ and $f \in \overline{co}\, D$ so that f is a weak* limit of convex combinations of members of D. The assertion follows from Lemma 4.12.

It would be interesting to examine the converse of Theorem 4.11. Arguments in [16, Theorem 3.4] fail essentially because $K(A \hat{\otimes}_\nu B)$ need not be equicontinuous, nor does the equality $A \hat{\otimes}_\nu B = A \hat{\otimes}_\alpha B$ seem to imply $\nu = \alpha$ automatically. However, it is possible to obtain a version of Theorem 4.11 that admits a converse, and that too can be regarded as a generalization of [16, Theorem 3.4].

Theorem 4.13. *For pro-C*-algebra A, the following are equivalent.*

(1) *A is nuclear.*
(2) *For every C*-algebra B, for every continuous complete state map* $\phi: A \to B^*$ *and for every p in* $S(A, \phi) = \{p \in S(A) \mid$ *there exists* $K > 0$ *such that* $\|\phi(x)\| \leqslant K p(x)$ *for all* $x\}$, *there exists a net* (ϕ_j) *of continuous complete state maps* $\phi_j: A \to B^*$ *of finite ranks such that*
(a) $\phi = \lim_j \phi_j$ *in simple weak* convergence,*
(b) $p \in S(A, \phi_j)$ *for all j.*

This can be proved by passing to the C^*-algebra quotient A_p and applying corresponding result for C^*-algebras.

5. An application: Multipliers of the Pedersen ideal of a C^*-algebra

A *multiplier* on a *algebra A without identity is a pair (l, r) consisting of linear maps $l, r: A \to A$ such that for all x, y in A; $l(xy) = l(x)y$, $r(xy) = xr(y)$ and $xl(y) = r(x)y$. The collection $\Gamma(A)$ of all multipliers on A is a *algebra with operations: $(l_1, r_1) + (l_2, r_2) = (l_1 + l_2, r_1 + r_2)$, $\lambda(l, r) = (\lambda l, \lambda r)$, $(l_1, r_1)(l_2, r_2) = (l_1 l_2, r_2 r_1)$ and $(l, r)^* = (r^*, l^*)$ where $r^*(a) = r(a^*)^*$, $l^*(a) = l(a^*)^*$. A is embedded as a two-sided *ideal in $\Gamma(A)$ via the *isomorphism $\mu: a \in A \to (l_a, r_a) \in \Gamma(A)$, $l_a x = ax$, $r_a x = xa$; and μ is onto iff A has identity. For a topological *algebra A, let $M(A) = \{(l, r) \in \Gamma(A) \mid l$ and r are continuous$\}$. If A is a pro-C^*-algebra, then $M(A) = \Gamma(A)$ [30]; and then $M(A)$ is a pro-C^*-algebra with *seminorm topology* τ defined by the calibration $\{\|\cdot\|_p \mid p \in S(A)\}$, $\|(l, r)\|_p = \sup\{p(l(a)) \mid p(a) \leqslant 1\}$ [23]. ($M(A)$ is a C^*-algebra iff A is a C^*-algebra) Corollary 4.8 (2) implies that if $M(A)$ is nuclear, then A is nuclear; but the converse does not hold even if A is a C^*-algebra. Take $A = K(H)$, the C^*-algebra of all compact operators on a Hilbert space H. Being a C^*-algebra of type I, $K(H)$ is nuclear; but $M(A) = B(H)$, (the C^*-algebra of all bounded operators on H) is not nuclear, for the Calkin algebra $B(H)/K(H)$ is known not to be nuclear.

Theorem 5.1. *Let A be a pro-C*-algebra. The C*-algebra* $b(M(A))$ *is isometrically *isomorphic to the C*-algebra* $M(b(A))$. *Thus* $M(A)$ *is a nuclear pro-C*-algebra iff* $b(A)$ *and the generalized Calkin algebra* $M(b(A))/b(A)$ *are nuclear C*-algebras.*

Proof. In view of Corollary 4.8 (2), it is sufficient to show that $b(M(A)) = M(b(A))$ up to isomorphism. Let (e_λ), $\|e_\lambda\| \leqslant 1$ be an approximate identity for the C^*-algebra $b(A)$. Then (e_λ) is also an approximate identity for A [23]. We show that $b(M(A)) \subset M(b(A))$. Let $(l,r) \in M(A)$ with $\|(l,r)\|_\infty = \sup\{\|(l,r)\|_p | p \in S(A)\} < \infty$. It is sufficient to show that $l(b(A)) \subset b(A)$, $r(b(A)) \subset b(A)$. For $x \in b(A)$, $p \in S(A)$, $p(l(x)) = p(l(\lim_{\lambda,\tau} e_\lambda x)) = p(\lim_{\lambda,\tau}(l(e_\lambda x))) = \lim_\lambda p(l(e_\lambda x)) = \lim_\lambda p(l(e_\lambda)x) \leqslant \lim_\lambda p(l(e_\lambda))p(x) \leqslant \|(l,r)\|_\infty p(x) \leqslant \|(l,r)\|_\infty \|x\|_\infty$ showing $l(x) \in b(A)$. Similarly $r(x) \in b(A)$. This defines a *homomorphism $\phi: b(M(A)) \to M(b(A))$, $\phi(l,r) = (l_{|b(A)}, r_{|b(A)})$. Since $b(A)$ is dense in A, ϕ is one–one. We show that ϕ is surjective by establishing $M(b(A)) \subset b(M(A))$ in the sense that given $(l,r) \in M(b(A))$ each of l and r extends uniquely as continuous linear maps $L, R: A \to A$ such that $(L,R) \in M(A)$, $\|(L,R)\|_\infty < \infty$. It is sufficient to show that each $l, r: (b(A),\tau) \to (b(A),\tau)$ is continuous, where τ is the relativization of the pro-C^*-topology from A. Since l, r are continuous in the C^*-topology on $b(A)$, $M_l = \sup\|l(e_\lambda)\|_\infty$, $M_r = \sup\|r(e_\lambda)\|_\infty < \infty$. Take $M = \{M_l, M_r\}$. Let $p \in S(A)$, $a \in b(A)$. Then $\|ae_\lambda - a\|_\infty \to 0$, $\|e_\lambda a - a\|_\infty \to 0$; and $p(l(a)) = \lim p(l(e_\lambda a))$ as $\|\cdot\|_\infty$-convergence implies τ-convergence. Thus $p(l(a)) = \lim p(l(e_\lambda)a) \leqslant \lim p(l(e_\lambda))p(a) \leqslant \sup p(l(e_\lambda))p(a) \leqslant Mp(a)$ giving the desired continuity of l (and of r, by a similar argument). This gives existence of continuous linear extensions $L, R: A \to A$. That $(L,R) \in M(A)$ is a consequence of density of $b(A)$ in A and joint continuity of multiplication in (A,τ), which also implies that for each $x \in A$, $p(L(x)) \leqslant Mp(x)$, $p(R(x)) \leqslant Mp(x)$ for all $p \in S(A)$. Hence $\|(L,R)\|_\infty = \sup_{p \in S(A)} \cdot \|(L,R)\|_p < \infty$ giving $(L,R) \in b(M(A))$. This completes the proof.

Now let A be a C^*-algebra. Let $\mathscr{K}_A$ denote the Pedersen ideal of A [22, p. 175], [21]. Let X be the primitive ideal space of A. Let $\mathscr{F}$ be the collection of all compact closed subsets of X. For an open subset U of X, let $I(U)$ be the closed ideal corresponding to U. It is shown in [25, Theorem 7] that $\Gamma(\mathscr{K}_A)$ is a pro-C^*-algebra realized as the multiplier algebra $M(B)$ of the pro-C^*-algebra $B = \varprojlim_{C \in \mathscr{F}} A/I(X - C)$; and $b(\Gamma(\mathscr{K})) = M(A)$ [25, Theorem 2]. Theorem 4.1 gives the following.

COROLLARY 5.2

For a C^*-algebra A, $\Gamma(\mathscr{K}_A)$ is a nuclear pro-C^*-algebra iff $M(A)$ is a nuclear C^*-algebra iff A and $M(A)/A$ are nuclear C^*-algebra.

Thus nuclearity of A is not sufficient to guarantee nuclearity of $\Gamma(\mathscr{K}_A)$. In the case of $A = K(H)$, one has: $\mathscr{K}_A =$ all finite rank operators in $B(H)$ and $\Gamma(\mathscr{K}_A) = M(A) = B(H)$ [18, p. 30].

6. Nuclear pro-C^*-algebras and linear topological nuclearity

If a pro-C^*-algebra is nuclear as a locally convex space in the sense of [26, Chapter IV], then we call A *linearly nuclear*, in which case, for every locally convex space $B, A \hat{\otimes}_\varepsilon B = A \hat{\otimes}_\pi B$ and $\varepsilon = \pi$. Thus a linearly nuclear pro-C^*-algebra is a nuclear pro-C^*-algebra. An infinite dimensional nuclear C^*-algebra fails to be linearly nuclear in view of Dvoretzky-Roger Theorem [26, Corollary 3, p. 184].

Theorem 6.1. *Let A be a σ-C^*-algebra. The following are equivalent.*

(1) *A is linearly nuclear.*

(2) A *is homeomorphically *isomorphic to an inverse limit of finite dim* *C^*-algebras. Further, if A is abelian and linearly nuclear, then A is homeomo* **isomorphic to the σ-C^*-algebra ω of all complex sequences with pointwise* *operations, complex conjugation of sequences as involution and the topology of p* *convergence.*

Proof. $(2)\Rightarrow(1)$ is a consequence of the fact that the inverse limit of nuclear nuclear. For $(1)\Rightarrow(2)$, let $P=(p_n)\subset S(A)$ be a sequence of seminorms determi topology t of A so that $A=\varprojlim_n A_n$; and by [23, Corollary 5.4], each $A_n=$ with the quotient topology t_q is a σ-C^*-algebra. The inclusion $id:(A_n,t_q)\to($ being automatically continuous [23, Theorem 5.2], it is homeomorphism mapping theorem; and hence the topology t_q on A_n is normable. Since a qu a nuclear space is nuclear, (A_n,t_q) is nuclear; and hence is finite dimensional Banach space. Thus (2) follows. Now let A be abelian and linearly nuclear. $A_n=k_n$, so that the C^*-algebra A_n is identified with $\mathbb{C}^{k_n}$. By [23, Propositio is *isomorphic to the abelian pro-C^*-algebra $C(X)$ of all continuous functi countably compactly generated Hausdorff space X. By above, X is count the sets $X_n=\{1,2,..k_n\}$ (= discrete Gelfand space of A_n) determine the top X (in the sense that $C\subset X$ is closed iff $C\cap X_n$ is closed for all n). Thus X is $X\approx\mathbb{N}$ and the assertion follows.

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A multiplier theorem for the sublaplacian on the Heisenberg group

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Abstract. A multiplier theorem for the sublaplacian on the Heisenberg group is proved using Littlewood–Paley–Stein theory of g-functions.

Keywords. Multiplier theorem; sublaplacian; Heisenberg group; Littlewood–Paley–Stein theory; g-functions.

1. Introduction

Consider the Heisenberg group H_n and the sublaplacian $\mathscr{L}$ on H_n. $\mathscr{L}$ is a formally non-negative hypoelliptic differential operator which has a unique self-adjoint extension to $L^2(H_n)$. If φ is a function defined on $\mathbb{R}$ then using spectral theorem one can define the operator $\varphi(\mathscr{L})$. If φ is a bounded function, then $\varphi(\mathscr{L})$ will be bounded on $L^2(H_n)$. In the same spirit one likes to find sufficient conditions on φ so that the operator $\varphi(\mathscr{L})$ will be bounded on $L^p(H_n)$.

This problem was studied by Mauceri [4] and the following result was proved.

If the function φ is $n+3$ times differentiable and satisfies the estimate $|\varphi^{(k)}(t)| \leqslant C(1+|t|)^{-k}, k=0,1,\ldots(n+3)$, then $\varphi(\mathscr{L})$ is bounded operator on $L^p(H_n)$ for all $1<p<\infty$.

This result was proved using the theory of singular integrals on homogeneous spaces developed by Coifman and Weiss [1]. Later Mauceri improved the above result replacing the smoothness condition on φ by a fractional order condition of the order $s>n+2$ (see [5]). Here we propose to give a different proof of the multiplier theorem. We prove:

Theorem. *Let φ be ν times differentiable and satisfies $|\varphi^{(k)}(t)| \leqslant C(1+|t|)^{-k}$ for $k=0,1,\ldots\nu$ where $\nu=n+2$ if n is even and $\nu=n+3$ if n is odd. Then $\varphi(\mathscr{L})$ is a bounded operator on $L^p(H_n)$, $1<p<\infty$.*

Our proof of this theorem is based on Littlewood–Paley–Stein theory of g_k and g_k^* functions. We adapt this method which was originally employed by Stein [6] to prove the Hormander–Mihlin multiplier theorem for the Fourier transform, to the present case. The same technique was successfully employed by Strichartz [7] and by the author [9], [10] to prove some multiplier theorems. One good thing about this approach is that the proof is simple and also we get a sharper result when n is even.

2. Preliminaries

The main reference for this section is [3]. See also [4]. The $(2n+1)$-dimensional Heisenberg group H_n is the nil potent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. The group structure is given by

$$(z,t)(\xi,s) = (z+\xi, t+s+2\,\mathrm{Im}\, z\cdot\bar{\xi}) \tag{1}$$

where $t, s\in\mathbb{R}$ and $z,\xi\in\mathbb{C}^n$. The Haar measure on H_n is simply the Lebesgue measure $dz ds$ on $\mathbb{C}^n\times\mathbb{R}$. For $w=(z,s)$ the homogeneous norm $|w|$ is defined by $|w|^4 = |z|^4 + s^2$.

We next recall the definition of the Fourier transform on H_n. The infinite dimensional representations of H_n are parametrized by $\mathbb{R}\backslash\{0\}$. If $\lambda\neq 0$, then all the representations π_λ can be realized on the same Hilbert space $L^2(\mathbb{R}^n)$. For $(z,s)\in H_n$, $\pi_\lambda(z,s)$ is the operator acting on $L^2(\mathbb{R}^n)$ by the prescription

$$\pi_\lambda(z,s)\varphi(\xi) = \exp(i\lambda s)\exp[i2\lambda(2\xi - x)\cdot y]\varphi(\xi - x), \tag{2}$$

where $z = x+iy$ and $\xi\in\mathbb{R}^n$.

The Fourier transform $\hat{f}$ of an L^1 function f on H_n is then the operator valued function

$$\hat{f}(\lambda) = \int_{H_n} f(w)\,\pi_\lambda(w)\,dw. \tag{3}$$

Then we have the following Plancherel formula:

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}}\int |\lambda|^n\,\|\hat{f}(\lambda)\|_{HS}^2\,d\lambda, \tag{4}$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. We also have an inversion formula

$$f(w) = \int \mathrm{tr}(\pi_\lambda(w)^*\hat{f}(\lambda))|\lambda|^n\,d\lambda, \tag{5}$$

where tr is the canonical semifinite trace.

For each $\lambda\neq 0$ we can select an orthonormal basis for $L^2(\mathbb{R}^n)$. Let $\Phi_\alpha^\lambda(x) = (2|\lambda|^{\frac{1}{2}})^{n/2}\,\Phi_\alpha((2|\lambda|^{\frac{1}{2}}x))$ where Φ_α are the Hermite functions on $\mathbb{R}^n$.Then $\{\Phi_\alpha^\lambda\}$ is an orthonormal basis for $L^2\{\mathbb{R}^n\}$. Let $P_N(\lambda)$ denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha^\lambda : |\alpha| = N\}$. Using these operators $P_N(\lambda)$ we can write the Fourier transform of a zonal function in a simple way.

Let $f(z,s) = f(|z|,s)$ be a zonal function and $\tilde{f}(z,\lambda)$ be the Fourier transform in the s-variable.

$$\tilde{f}(z,\lambda) = \int \exp(i\lambda s) f(z,s)\,ds. \tag{6}$$

Define $R_N(\lambda, f)$ by the formula

$$R_N(\lambda,f) = C_N\frac{N!}{(N+n-1)!}\int_0^\infty \tilde{f}(r,\lambda)\,L_N^{n-1}(2|\lambda|r^2)\exp(-|\lambda|r^2)r^{2n-1}\,dr, \tag{7}$$

where L_N^{n-1} are the Laguerre polynomials of type $(n-1)$. Then one has

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f) P_N(\lambda). \tag{8}$$

And the Plancherel formula takes the form

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int \sum_{N=0}^{\infty} |R_N(\lambda, f)|^2 \frac{(N+n-1)!}{N!} |\lambda|^n \, \mathrm{d}\lambda. \tag{9}$$

On H_n consider the following left invariant vector fields.

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \; \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}. \tag{10}$$

The sublaplacian $\mathscr{L}$ is then defined by

$$\mathscr{L} = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j). \tag{11}$$

Each representation π_λ determines a Lie algebra representation $\mathrm{d}\pi_\lambda$. It can be shown that $\mathrm{d}\pi_\lambda(\mathscr{L})$ is a closable operator. Its closure is denoted by $H(\lambda)$ and it has the following spectral decomposition:

$$H(\lambda) = \sum_{N=0}^{\infty} (2N+n)|\lambda| P_N(\lambda). \tag{12}$$

For any reasonable function φ on $\mathbb{R}$, using spectral theorem, one can define the operator $\varphi(\mathscr{L})$. It can be shown that $\varphi(\mathscr{L})$ is a convolution operator with kernel k i.e. $\varphi(\mathscr{L})f = k * f$. The Fourier transform of k is given by

$$\hat{k}(\lambda) = \sum_{N=0}^{\infty} \varphi((2N+n)|\lambda|) P_N(\lambda). \tag{14}$$

All these things will be made use of in the following sections.

3. Littlewood–Paley–Stein theory on H_n

In [2] Folland has shown that the sublaplacian $\mathscr{L}$ generates a contraction semigroup T^t which satisfies all the conditions required to develop a Littlewood–Paley–Stein theory (see [6]). As in Stein [6] we define, for each positive integer k, the following functions

$$(g_k(f, w))^2 = \int_0^{\infty} t^{2k-1} |\partial_t^k T^t f(w)|^2 \, \mathrm{d}t \tag{15}$$

$$(g_k^*(f, w))^2 = \int_{H_n} \int_0^{\infty} t^{-n} (1 + t^{-2}|v|^4)^{-k} |\partial_t^k T^t f(v^{-1}w)|^2 \, \mathrm{d}t \, \mathrm{d}v. \tag{16}$$

For these functions we will prove the following theorem.

Theorem 3.1. (i) *For* $k \geqslant 1$, $\|g_k(f)\|_2 = 2^{-k}\|f\|_2$.
(ii) *For* $1 < p < \infty$, $C_1\|f\|_p \leqslant \|g_k(f)\|_p \leqslant C_2\|f\|_p$.
(iii) *If* $k > (n+1)/2$ *and* $p > 2$, *then* $\|g_k^*(f)\|_p \leqslant C\|f\|_p$.

Proof. The inequality $\|g_k(f)\|_p \leqslant C_2\|f\|_p$ follows from the general theory. The reverse inequality can be easily deduced once we have (i). When $k > (n+1)/2$, the function $(1+|v|^4)^{-k}$ is integrable and hence one can prove (iii) using (i). This is routine and well known. So, it remains to prove (i).

We prove (i) when $k = 1$. The case $k > 1$ is similar. From the definition it follows that

$$\|g_1(f)\|_2^2 = \int_0^\infty \int_{H_n} t|\partial_t T^t f(w)|^2 \mathrm{d}w\,\mathrm{d}t. \tag{17}$$

In view of the Plancherel formula (4) the integral becomes

$$\int_{H_n} |\partial_t T^t f(w)|^2 \mathrm{d}w = \frac{2^{n-1}}{\pi^{n-1}} \int |\lambda|^n \|(\partial_t T^t f)\hat{}(\lambda)\|_{HS}^2 \mathrm{d}\lambda. \tag{18}$$

Since $T^t f = \exp(-t\mathscr{L})f$, we see that

$$(\partial_t T^t f)\hat{}(\lambda) = -H(\lambda)\exp(-tH(\lambda))\hat{f}(\lambda) \tag{19}$$

and hence its squared Hilbert-Schmidt norm is given by the expression

$$\sum_\alpha ((2|\alpha|+n)|\lambda|)^2 \exp(-2t(2|\alpha|+n)|\lambda|)(\Phi_\alpha^\lambda, \hat{f}(\lambda)^*\hat{f}(\lambda)\Phi_\alpha^\lambda). \tag{20}$$

If we use this in (18) and integrate with respect to $t\,\mathrm{d}t$, we will get

$$\|g_1(f)\|_2^2 = 2^{-2}\frac{2^{n-1}}{\pi^{n+1}} \int |\lambda|^n \|\hat{f}(\lambda)\|_{HS}^2 \mathrm{d}\lambda.$$

And this proves that $\|g_1(f)\|_2 = 2^{-1}\|f\|_2$.

4. The multiplier theorem

Let us set $Mf = \varphi(\mathscr{L})f$. To prove the multiplier theorem what we need is the following pointwise inequality.

$$g_{k+1}(Mf) \leqslant C g_k^*(f) \tag{21}$$

for some integer $k > (n+1)/2$. For then the multiplier theorem for $p > 2$ will follow immediately from Theorem 3.1. For $p < 2$ one can use duality to conclude that M is bounded on $L^p(H_n)$.

So, we proceed to prove the inequality (21). Let us set $u_t = T^t f$, $U_t = T^t(Mf)$. Then it is easy to see that

$$U_{t+s}(w) = (G_t * u_s)(w) \tag{22}$$

where the Fourier transform of G_t is given by

$$\hat{G}_t(\lambda) = \sum_{N=0}^\infty \exp(-(2N+n)|\lambda|t)\varphi((2N+n)|\lambda|)P_N(\lambda). \tag{23}$$

tiating (22) k times with respect to t and once with respect to s and putting obtain

$$\partial_t^{k+1} T^{2t}(Mf) = F_t * \partial_t T^t f, \tag{24}$$

ie Fourier transform of F_t is given by

$$\hat{F}_t(\lambda) = (-1)^k \sum_{N=0}^{\infty} \exp(-(2N+n)|\lambda|t)(2N+n)^k |\lambda|^k \varphi((2N+n)|\lambda|) P_N(\lambda). \tag{25}$$

:e, we have

$$|\partial_t^{k+1} T^{2t}(Mf)(w)| \leqslant \int |F_t(v)| |\partial_t T^t f(v^{-1}w)| \, \mathrm{d}v.$$

g Cauchy–Schwartz inequality

$$|\partial_t^{k+1} T^{2t}(Mf)(w)|^2 \leqslant A_t \cdot B_t(w), \tag{26}$$

e have written

$$A_t = \int |F_t(v)|^2 (1 + t^{-2}|v|^4)^k \, \mathrm{d}v$$

$$B_t(w) = \int (1 + t^{-2}|v|^4)^{-k} |\partial_t T^t(v^{-1}w)|^2 \, \mathrm{d}v. \tag{27}$$

complete the proof we need the estimate of the following Lemma.

Under the hypothesis of the theorem the estimate $A_t \leqslant C\, t^{-n-2k-1}$ is valid :s the smallest integer greater than $(n+1)/2$.

ning the lemma for a moment it is easy to establish inequality (21). Indeed,) we have

$$|\partial_t^{k+1} T^{2t}(Mf)(w)|^2 \leqslant C t^{-n-2k-1} B_t(w).$$

ing this against t^{2k+1} we get

$$g_{k+1}(Mf, w) \leqslant C g_k^*(f, w).$$

npletes the proof of the multiplier theorem modulo the above lemma.

f of the Lemma

e the Lemma let us write

$$I = \int_{|w| \leqslant \sqrt{t}} |F_t(w)|^2 (1 + t^{-2}|w|^4)^k \, \mathrm{d}w \tag{28}$$

$$J = \int_{|w| > \sqrt{t}} |F_t(w)|^2 (1 + t^{-2}|w|^4)^k \, \mathrm{d}w. \tag{29}$$

Estimating the integral I is easy. We note that since $|w| \leqslant \sqrt{t}$

$$I \leqslant C \int |F_t(w)|^2 \mathrm{d}w$$

and hence in view of Plancherel formula

$$I \leqslant C \int |\lambda|^n \left(\sum_{N=0}^{\infty} (2N+n)^{2k} |\lambda|^{2k} \exp[-2|\lambda|(2N+n)t] \frac{(N+n-1)!}{N!} \right) \mathrm{d}\lambda$$
$$\leqslant Ct^{-n-2k-1} (\Sigma(2N+n)^{-2}) \leqslant Ct^{-n-2k-1}.$$

This proves the estimate for the integral I.

Next consider J. Let us write $w = (z, s)$. We observe that

$$J \leqslant Ct^{-2k} \int\int (s^2 + |z|^4)^k |F_t(z,s)|^2 \mathrm{d}z\,\mathrm{d}s$$
$$= Ct^{-2k} \int\int |(is - |z|^2)^k F_t(z,s)|^2 \mathrm{d}z\,\mathrm{d}s. \tag{30}$$

If we can show that the integral in (30) is bounded by t^{-n-1} then we are done. If we write the Fourier transform of $G = (is - r^2)^k F_t(z,s)$ in the form

$$\hat{G}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, (is - |z|^2)^k F_t) P_N(\lambda)$$

then we need to show that

$$\int \sum_{N=0}^{\infty} |R_N(\lambda, (is - r^2)^k F_t)|^2 \frac{(N+n-1)!}{N!} |\lambda|^n \mathrm{d}\lambda \leqslant Ct^{-n-1} \tag{31}$$

where we have set $|z|^2 = r^2$.

Let us write

$$\psi(N,\lambda) = (-1)^k (2N+n)^k |\lambda|^k \exp[-(2N+n)|\lambda| t] \varphi((2N+n)|\lambda|)$$

so that $R_N(\lambda, F_t) = \psi(N, \lambda)$. We define $\psi_k(N,\lambda)$ to be $R_N(\lambda, (is - r^2)^k F_t)$. Then the following estimate is valid.

Lemma 5.1. *Under the hypothesis of the theorem there is an* $\varepsilon > 0$ *such that*

$$|\psi_k(N,\lambda)| \leqslant C \exp[-\varepsilon(2N+n)|\lambda| t. \tag{32}$$

If we use (32) in (29) then the estimate $J \leqslant t^{-n-2k-1}$ is immediate. So we proceed to prove Lemma 5.1.

Recall the definition of $R_N(\lambda, f)$ for a zonal function f.

$$R_N(\lambda, f) = C_n \frac{N!}{(N+n-1)!} \int_0^{\infty} \tilde{f}(r,\lambda)\, L_N^{n-1}(2|\lambda| r^2) \exp(-|\lambda| r^2) r^{2n-1} \mathrm{d}r, \tag{33}$$

where $\tilde{f}(r,\lambda)$ is the Euclidean Fourier transform of f in the s variable. We will prove

2) when $\lambda > 0$. The case $\lambda < 0$ is completely similar.
Since $(isf)\tilde{}(r,\lambda) = (\mathrm{d}/\mathrm{d}\lambda)\tilde{f}(r,\lambda)$ we obtain

$$R_N(\lambda, isf) = \frac{\mathrm{d}}{\mathrm{d}\lambda} R_N(\lambda, f) - C_n \frac{N!}{(N+n-1)!} \int_0^\infty \tilde{f}(r,\lambda) \times \frac{\mathrm{d}}{\mathrm{d}\lambda}\{L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2)\} r^{2n-1}\,\mathrm{d}r.$$

ow

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2)$$
$$= 2r^2 \frac{\mathrm{d}}{\mathrm{d}r} L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2) - r^2 L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2).$$

ing the recursion formula (see [8])

$$r\frac{\mathrm{d}}{\mathrm{d}r} L_N^{n-1}(r) = N L_N^{n-1}(r) - (N+n-1) L_{N-1}^{n-1}(r) \tag{34}$$

imple calculation shows that

$$R_N(\lambda, isf) = \frac{\mathrm{d}}{\mathrm{d}\lambda} R_N(\lambda, f) - \frac{N}{\lambda}(R_N(\lambda, f) - R_{N-1}(\lambda, f)) + R_N(\lambda, r^2 f).$$

us we have obtained the formula

$$\psi_1(N,\lambda) = \frac{\partial\psi}{\partial\lambda} - \frac{N}{\lambda}(\psi(N,\lambda) - \psi(N-1,\lambda)). \tag{35}$$

ice $\psi(N,\lambda) = \psi((2N+n)\lambda)$ we can write (35) in the form

$$\psi_1(N,\lambda) = \frac{1}{2}\frac{n}{\lambda}\frac{\partial\psi}{\partial N} + \frac{N}{\lambda}\left(\frac{\partial\psi}{\partial N} - \Delta\psi\right), \tag{36}$$

ere $\Delta\psi(N,\lambda) = \psi(N,\lambda) - \psi(N-1,\lambda)$. Define the operators S, D and T by

$$S\psi = \frac{\partial\psi}{\partial N}, \quad D\psi = \frac{\partial\psi}{\partial N} - \Delta\psi, \quad T\psi = ND\psi.$$

, we have

$$\psi_1(N,\lambda) = \lambda^{-1}\left(\frac{n}{2}S + T\right)\psi(N,\lambda). \tag{37}$$

om this formula we can conclude that

$$\psi_k(N,\lambda) = \lambda^{-k} \sum_{i+j+m=k} a_{ijm} S^i T^j S^m \psi(N,\lambda). \tag{38}$$

Now we observe that $S^m\psi(N,\lambda) = \psi^{(m)}((2N+n)\lambda)(2\lambda)^m$ and by hypothesis of the eorem S^m in essence brings a factor $(2N+n)^{-m}$. We will show that T^j also does

the same thing. Then each term in the sum (38) will behave like $\lambda^{-k}(2N+n)^{-k}\psi(N,\lambda)$. Recalling the definition of $\psi(N,\lambda)$ we see that

$$|\psi_k(N,\lambda)| \leqslant C\exp[-\varepsilon(2N+n)\lambda t]$$

as desired.

For the operators T^j the following formula is valid.

Lemma 5.2.

$$T^j\psi = \sum C_{pqm} N^p D^q(\Delta^m\psi)$$

where the sum is extended over all p, q, m satisfying the relation $j+p \leqslant 2q+m \leqslant 2j$.

Proof. We prove this lemma by induction. We first observe that from the definition of T, the lemma is trivially valid for $j=1$. Now assume the lemma true for some j and consider $T^{j+1}\psi$

$$T^{j+1}\psi = \sum C_{pqm} ND(N^p D^q(\Delta^m\psi)) \tag{39}$$

where $j+p \leqslant 2q+m \leqslant 2j$. We need a formula for $D(N^p D\psi)$.

We claim that

$$D(N^p D\psi) = N^p D^2\psi + \sum_{i=0}^{p-1} a_i N^i D(\Delta\psi) + \sum_{i=0}^{p-2} b_i N^i D\psi. \tag{40}$$

Assuming the claim for a moment we have

$$T^{j+1}\psi = \sum_{p,q,m} C_{pqm} N^{p+1} D^{q+1}(\Delta^m\psi) + \sum_{p,q,m} C_{pqm} \sum_{i=0}^{p-1} a_i N^{i+1} D^q(\Delta^{m+1}\psi)$$
$$+ \sum_{p,q,m} C_{pqm} \sum_{i=0}^{p-2} b_i N^{i+1} D^q(\Delta^m\psi).$$

From this formula it is clear that $T^{j+1}\psi$ is of the desired form.

To prove the claim we first observe that

$$\Delta(\varphi\psi)(N) = \Delta\varphi(N)\psi(N) + \varphi(N-1)\Delta\psi(N). \tag{41}$$

In view of this formula

$$\Delta(N^p D\psi) = \Delta(N^p)D\psi + (N-1)^p D(\Delta\psi). \tag{42}$$

We also have

$$\Delta(N^p) = N^p - (N-1)^p = pN^{p-1} - \sum_{i=0}^{p-2} b_i N^i \tag{43}$$

$$(N-1)^p D(\Delta\psi) = N^p D(\Delta\psi) - \sum_{i=0}^{p-1} a_i N^i D(\Delta\psi) \tag{44}$$

$$\frac{\partial}{\partial N}(N^p D\psi) = pN^{p-1}D\psi + N^p D\left(\frac{\partial\psi}{\partial N}\right). \tag{45}$$

rom (42)–(45) it follows that

$$D(N^p D\psi) = N^p D^2 \psi + \sum_{i=0}^{p-1} a_i N^i D(\Delta\psi) + \sum_{i=0}^{p-2} b_i N^i D\psi. \tag{46}$$

his proves the claim.

Finally we will show that the action of T^j has the desired properties. We have

$$T^j \psi = \sum C_{pqm} N^p D^q (\Delta^m \psi), \tag{47}$$

here $p + j \leqslant 2q + m \leqslant 2j$. Now using Taylor's formula with integral form of remainder e can write

$$D\psi(N) = \int_0^1 t\psi''(N - 1 + t, \lambda)\,\mathrm{d}t, \tag{48}$$

here the primes stand for the derivatives with respect to N. From (48) it is clear iat the action of D is to bring down the factor N^{-2}. An iteration will show that D^q ill bring down the factor of N^{-2q} when applied to ψ. Since $\Delta^m \psi$ brings down N^{-m} ie formula (47) shows that T^j acting on ψ brings down the factor

$$\sum C_{pqm} N^p N^{-2q-m}.$$

nce $p + j \leqslant 2q + m$, essentially T^j brings down a factor of N^{-j} as required.

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A note on the multidimensional Weyl fractional operator

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Abstract. The purpose of the present paper is to establish a connection theorem involving the multidimensional Weyl fractional operator and the classical multidimensional Laplace transform. This provides an extension of a result due to Raina and Koul [6].

Keywords. Multidimensional Weyl operator; multidimensional Laplace transform; differintegral operators.

1. Introduction

In the theory of familiar Weyl fractional calculus, Raina and Koul [6] established a connection theorem involving the Laplace transform of $t^q f(t)$ for arbitrary (real) q. With a view to generalizing this Raina–Koul result [6, p. 180, eq. (6)], we try to establish its multidimensional extension (in a slightly variant form). We invoke in our analysis the multidimensional Weyl fractional operator defined and introduced quite recently by Srivastava and Raina [10].

2. Preliminaries and definitions

In the literature there are numerous examples of operators of fractional differintegrals (that is, fractional derivatives and fractional integrals) in a wide variety of fields (see, for example, [3], [4], [7], [8], [9]). Much of the theory of fractional calculus is based upon the familiar differintegral operator ${}_cD_z^\mu$ defined by ([2] and [7])

$$
{}_cD_z^\mu f(z) = \frac{1}{\Gamma(m-\mu)} \frac{\mathrm{d}^m}{\mathrm{d}z^m} \int_c^z (z-t)^{m-\mu-1} f(t)\,\mathrm{d}t
$$

$$
(m > \mathrm{Re}(\mu); \quad m \in N_0 = N \cup \{0\}; \quad N = \{1, 2, 3, \ldots\}). \tag{1}
$$

For $c = 0$, eq. (1) defines the classical Riemann–Liouville fractional derivative (or integral) of order μ (or $-\mu$). On the other hand, when $c \to \infty$, (1) may be identified with the definition of the familiar Weyl fractional derivative (or integral) of order μ (or $-\mu$) (see Erdélyi *et al* [1, Vol. II, Chapter 13] for details). By repeated applications of the operator ${}_cD_z^\mu$ to a given function of several variables (see Raina [5]), a corresponding multidimensional fractional operator can be defined in a natural way. Indeed, Srivastava and Raina [10] developed the corresponding multidimensional fractional derivative (or integral) operator. In particular, the corresponding multi-

dimensional extension of Weyl operator of fractional calculus is defined (see [10] for details)

$$
\begin{aligned}
&W_{\mu_1,\ldots,\mu_n} f(z_1,\ldots,z_n)\\
&=\frac{(-1)^{m_1+\cdots+m_n}}{\Gamma(m_1-\mu_1)\cdots\Gamma(m_n-\mu_n)}\frac{\partial^{m_1+\cdots+m_n}}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}\int_{z_1}^{\infty}\cdots\int_{z_n}^{\infty} f(t_1,\ldots,t_n)\\
&\quad\times(t_1-z_1)^{m_1-\mu_1-1}\cdots(t_n-z_n)^{m_n-\mu_n-1}\,\mathrm{d}t_1\cdots\mathrm{d}t_n\\
&\quad(m_j>\mathrm{Re}(\mu_j);\quad m_j\in N_0;\quad j=1,\ldots,n).
\end{aligned}
\tag{2}
$$

Further, if $f(t_1,\ldots,t_n)$ is piecewise continuous for each $t_j\in[0,\infty)$, $j=1,\ldots,n$; and if

$$
|f(t_1,\ldots,t_n)|\leqslant M\exp(p_1t_1+\cdots+p_nt_n),
\tag{3}
$$

for all $t_j\geqslant T_j$ $(j=1,\ldots,n)$, M and T_j being positive constants, then the n-dimensional Laplace transform of $f(t_1,\ldots,t_n)$ is defined by

$$
\begin{aligned}
&L\{f(t_1,\ldots,t_n:s_1,\ldots,s_n)\}=F(s_1,\ldots,s_n)\\
&=\int_0^{\infty}\cdots\int_0^{\infty}\exp(-s_1t_1-\cdots-s_nt_n)f(t_1,\ldots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n,
\end{aligned}
\tag{4}
$$

where for convergence, $\mathrm{Re}(s_j-p_j)>0$; $j=1,\ldots,n$.

3. The main result

We propose to establish a result connecting the multidimensional Laplace transform (4) with the multidimensional fractional operator (2). Our result is contained in the following:

Theorem. *Let the Laplace transform of a function $f(t_1,\ldots,t_n)$ be defined by* (4). *Then*

$$
W_{q_1,\ldots,q_n}F(s_1,\ldots,s_n)=L\{t_1^{q_1}\cdots t_n^{q_n}f(t_1,\ldots,t_n):s_1,\ldots,s_n\},
\tag{5}
$$

holds true for all (real) values of $q_i(i=1,\ldots,n)$, provided that the Weyl operator exists.

Proof. In view of the Remarks (1 and 2) stated in pages 358 and 359, respectively, in [10], concerning the interpretations of the Weyl operator in the two cases $\mathrm{Re}(q_j)<0$, and $\mathrm{Re}(q_j)\geqslant 0$, for $j=1,\ldots,n$, the assertion (5) follows straightforwardly.

When $\mu_j\to m_j(m_j\in N_0; j=1,\ldots,n)$ in (2), and noting the relationship [10, p. 360, eq. (1.12)]:

$$
W_{m_1,\ldots,m_n}f(z_1,\ldots,z_n)=\frac{\partial^{m_1+\cdots+m_n}}{\partial z_1^{m_1}\ldots\partial z_n^{m_n}}f(z_1,\ldots,z_n),\quad m_j\in N_0;\quad j=1,\ldots,n;
\tag{6}
$$

then in this case, our result (5) gives the multidimensional extension of a result recorded in [8, p. 117, eq (7.19)] (and also in [11]).

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Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media

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Abstract. An analysis of the scattering of horizontally polarized shear wave by a semi-infinite crack running with uniform velocity along the interface of two dissimilar semi-infinite elastic media has been carried out. The mixed boundary value problem has been solved completely by the Wiener–Hopf technique. The effect of different values of the material parameter, the angle of incidence of incident wave and the crack propagation velocity on the stress intensity factor have been illustrated graphically.

Keywords. Diffraction of elastic waves; propagating crack; SH-wave; stress intensity factor.

1. Introduction

It is well known that the problems of diffraction of elastic wave by cracks or inclusions are of considerable importance in view of their application in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has also evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder *et al* [5] studied the diffraction of monochromatic plane SH-waves obliquely incident on a rigid half plane between the two different semi-infinite media.

In this paper we have considered the problem of the diffraction of a plane harmonic SH-wave by a semi-infinite crack running uniformly along the interface of two dissimilar semi-infinite elastic media. The problem of scattering of plane harmonic polarized shear wave by a half plane crack in an infinite isotropic medium extending under antiplane strain was studied earlier by Jahanshahi [3]. Chen and Sih [1, 2] also solved the in plane problem of the diffraction of stress waves by a running crack in an incident wave field in an infinite elastic medium. We have applied Fourier transform and Wiener–Hopf technique [4] to solve the mixed boundary value problem. The resulting integrals have been evaluated asymptotically to obtain the displacement and stress field near about the crack tip. It is found that the stress intensity factor depends sensitively upon the speed of crack propagation, the angle of incidence of the incoming wave and on the material properties of the elastic media. Quantitative assessment of the effect of the aforementioned parameters on the stress intensity factor has been made by displaying the numerical results graphically for two pairs of different materials.

2. Formulation of the problem and its solution

Let a plane crack move at a constant velocity V on the interface of two bonded dissimilar elastic semi-infinite medium due to the incidence of the plane harmonic SH-wave

$$v_1^0 = V_1 \exp[-i\{\Lambda_1(X\cos\Theta_1 + Y\sin\Theta_1) + \Omega T\}] \tag{1}$$

in the medium where the co-efficient of rigidity, density and shear-wave velocity respectively are given by μ_1, ρ_1 and C_1. The crack lies on the bimaterial interface along $Y=0$ with respect to the fixed rectangular co-ordinate system (X, Y, Z).

We assume that the displacement and stress due to the scattered field are

$$v_j = v_j(X, Y, T) \tag{2}$$

and

$$(\tau_{xz})_j = \mu_j \frac{\partial v_j}{\partial X}, \quad (\tau_{yz})_j = \mu_j \frac{\partial v_j}{\partial Y} \tag{3}$$

where the subscript $j=1,2$ refers to the upper and lower half-planes and T the time.

The equations of SH-wave motion in either elastic half-space are given by

$$\frac{\partial^2 v_j}{\partial X^2} + \frac{\partial^2 v_j}{\partial Y^2} = \frac{1}{C_j^2}\frac{\partial^2 v_j}{\partial T^2} \quad (j=1,2) \tag{4}$$

where $C_j = (\mu_j/\rho_j)^{1/2}$ is the shear-wave velocity. Without any loss of generality, we further assume that $C_1 > C_2$.

Due to the incident wave given in (1), the reflected and transmitted waves in the absence of the crack may be written in the form

$$v_1^r(X, Y, T) = V_1^r \exp[-i\{\Lambda_1(X\cos\Theta_1 - Y\sin\Theta_1) + \Omega T)\}]$$

and

$$v_2^T(X, Y, T) = V_2^T \exp[-i\{\Lambda_2(X\cos\Theta_2 + Y\sin\Theta_2) + \Omega T)\}] \tag{5}$$

where

$$V_1^r = \frac{\mu_1\Lambda_1\sin\Theta_1 - \mu_2\Lambda_2\sin\Theta_2}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_1^R V_1 \quad \text{(say)}$$

and

$$V_2^T = \frac{2\mu_1\Lambda_1\sin\Theta_1}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_2^T V_1 \quad \text{(say)} \tag{6}$$

with

$$\Lambda_1\cos\Theta_1 = \Lambda_2\cos\Theta_2.$$

V_1, V_1^r and V_2^T are the incident, reflected and transmitted wave amplitude respectively, Λ_j the wave number, $\Omega = \Lambda_j C_j$ the circular frequency and Θ_1, Θ_2 the angles of incidence and refraction respectively.

Assume that the crack has been moving in the horizontal direction along the interface for a sufficiently long time and that a steady state has been reached in the neighbourhood of the crack.

A set of moving co-ordinate systems (x, y, z, t) attached to the crack tip moving at

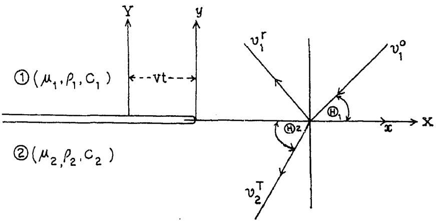

Figure 1. Geometry of the propagating crack.

a constant velocity V is introduced in accordance with

$$x = X - Vt, \quad y_j = s_j Y, \quad z = Z, \quad t = T \tag{7}$$

where $s_j = (1 - M_j^2)^{1/2}$ and $M_j = V/C_j$ is the Mach number.

In terms of the moving co-ordinate system (x, y, t), (4) becomes

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y_j^2} + \frac{1}{C_j^2 s_j^2}\frac{\partial}{\partial t}\left(2M_j C_j \frac{\partial v_j}{\partial x} - \frac{\partial v_j}{\partial t}\right) = 0. \tag{8}$$

It is convenient to define an apparent circular frequency $\omega = \alpha\Omega$ and the angles of reflection ϕ_1 and refraction ϕ_2 are given by

$$\cos\phi_j = M_j + (\Lambda_j/\lambda_j)\cos\Theta_j, \quad \sin\phi_j = (s_j/\alpha)\sin\Theta_j,$$

where

$$\alpha = (1 + M_j \cos\Theta_j) \quad \text{and} \quad \lambda_j = (\Lambda_j/s_j^2)\alpha. \tag{9}$$

Using these relations in a moving system, (1) and (5) take the form

$$\begin{bmatrix} v_1^0 \\ v_1^r \\ v_2^T \end{bmatrix} = \begin{bmatrix} w_1^0(x, y_1) \\ w_1^r(x, y_1) \\ w_2^T(x, y_2) \end{bmatrix} \exp\{i(M_1\lambda_1 x - \omega t)\} \tag{10}$$

where

$$\begin{aligned} w_1^0(x, y_1) &= V_1 \exp\{-i\lambda_1(x\cos\phi_1 + y_1\sin\phi_1)\} \\ w_1^r(x, y_1) &= A_1^R V_1 \exp\{-i\lambda_1(x\cos\phi_1 - y_1\sin\phi_1)\} \\ w_2^T(x, y_2) &= A_2^T V_1 \exp[-i\{(\beta_2 + \lambda_2\cos\phi_2)x + \lambda_2 y_2 \sin\phi_2\}] \end{aligned} \tag{11}$$

and

$$\beta_2 = M_1\lambda_1\left(1 - \frac{\lambda_2 C_1}{\lambda_1 C_2}\right) < 0 \quad \text{since} \quad C_1 > C_2.$$

Using (10), we assume the solution of the governing equation (8) as

$$v_j(x, y_j, t) = w_j(x, y_j)\exp[i(M_j\lambda_j x - \omega t)]. \tag{12}$$

Substitution of (12) in (8) yields the Helmholtz equation

$$\frac{\partial^2 w_j}{\partial x^2}+\frac{\partial^2 w_j}{\partial y_j^2}+\lambda_j^2 w_j=0 \quad (j=1,2). \tag{13}$$

Applying Fourier transform, (13) can be solved and the result is

$$w_1(x,y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\exp\{-i\xi x-(\xi^2-\lambda_1^2)^{1/2}y_1\}\,d\xi, \quad (y_1>0)$$

and

$$w_2(x,y_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(\xi)\exp\{-i\xi x-(\xi^2-\lambda_2^2)^{1/2}y_2\}\,d\xi, \quad (y_2<0) \tag{14}$$

where $A_1(\xi)$ and $A_2(\xi)$ are the unknown functions to be determined. From (12) and (14) we obtain the displacement components due to scattered field as

$$v_1=\exp[i(M_1\lambda_1 x-\omega t]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(u)\exp[-iux-\gamma_1 y_1]\,du, \quad (y_1>0)$$

and

$$v_2=\exp[i(M_1\lambda_1 x-\omega t]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(u)\exp[-iux+\gamma_2 y_2]\,du, \quad (y_2<0) \tag{15}$$

where

$$\gamma_1=(u^2-\lambda_1^2)^{1/2} \quad \text{and} \quad \gamma_2=[(u-\beta_2)^2-\lambda_2^2]^{1/2}. \tag{16}$$

Therefore, the expressions for the stresses are

$$(\tau_{xz})_1=-i\mu_1\exp[i(M_1\lambda_1 x-\omega t)]$$
$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_1(u)\exp[-iux-\gamma_1 y_1]\,du$$
$$(\tau_{xz})_2=-i\mu_2\exp[i(M_1\lambda_1 x-\omega t)]$$
$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_2(u)\exp[-iux+\gamma_2 y_2]\,du$$

and

$$(\tau_{y_1z})_1=-\mu_1 s_1\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_1 A_1(u)\exp[-iux-\gamma_1 y_1]\,du$$

$$(\tau_{y_2z})_2=\mu_2 s_2\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_2 A_2(u)\exp[-iux+\gamma_2 y_2]\,du. \tag{17}$$

The unknown functions $A_1(u)$ and $A_2(u)$ are to be determined from the following boundary conditions at the interface $y=0$

(i) $\quad v_1(x,0)=v_2(x,0), \quad x>0$

(ii) $\quad \mu_1 s_1\dfrac{\partial v_1}{\partial y_1}=\mu_2 s_2\dfrac{\partial v_2}{\partial y_2}, \quad -\infty<x<\infty$

and

(iii) $$\frac{\partial v_1^0}{\partial y_1}+\frac{\partial v_1^r}{\partial y_1}+\frac{\partial v_1}{\partial y_1}=0, \quad x<0, \quad y\to 0+.$$

From the boundary condition (ii) we obtain

$$\mu_1 s_1 \gamma_1 A_1(u)+\mu_2 s_2 \gamma_2 A_2(u)=0 \tag{18}$$

and from other two boundary conditions, we get

$$\int_{-\infty}^{\infty} B_1(u)\exp(-iux)\,\mathrm{d}u=0 \quad (x>0)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} M(u)B_1(u)\exp(-iux)\,\mathrm{d}u=N\exp[-i\lambda_1 x\cos\phi_1], \quad (x<0) \tag{19}$$

where

$$B_1(u)=\frac{\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2}{\mu_2 s_2\gamma_2}A_1(u)$$

$$M(u)=\gamma_1\frac{\mu_2 s_2\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)} \tag{20}$$

and

$$N=-\frac{i\Lambda_1 v_1\sin\Theta_1}{s_1}(1-A_1^R).$$

The solution of the dual integral equation may be obtained by a method based on the Wiener–Hopf technique. The first part of (19) can be satisfied if we choose

$$B_1(u)=L_-(u) \tag{21}$$

where $L_-(u)$ is a function of u, analytic in the lower half of the complex u-plane. The second part is satisfied if we take

$$M(u)B_1(u)=\frac{N}{i(u-\alpha_1)}\frac{U_+(u)}{U_+(\alpha_1)} \tag{22}$$

where $\alpha_1=\lambda_1\cos\phi_1$ and $U_+(u)$ is a function free from zeros and singularities in the upper half of the complex u-plane. Thus (22) is a solution of the second part of (19) can easily be shown by completing the path from $-\infty$ to ∞ by a semi-circle of infinite radius in the upper u-plane and then applying the residue theorem and Jordan's Lemma. The path of integration is chosen to avoid possible branch points and is indented below the pole $u=\alpha_1$.

Eliminating $B_1(u)$ from (21) and (22) we obtain

$$\frac{L_-(u)}{U_+(u)}=\frac{N}{i(u-\alpha_1)M(u)}\frac{1}{U_+(\alpha_1)} \tag{23}$$

and

$$M(u)=\frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2}(u^2-\lambda_1^2)^{1/2}F(u) \tag{24}$$

where

$$F(u)=\frac{(\mu_1 s_1+\mu_2 s_2)\gamma_2}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)}$$

and

$$F(u)\to 1 \quad \text{as} \quad |u|\to\infty.$$

The function $F(u)$ can be expressed as the product of two functions such that

$$F(u)=F_+(u)\cdot F_-(u) \tag{25}$$

where $F_+(u)$ and $F_-(u)$ are analytic in the upper and lower half of the complex u-plane respectively. The expressions for $F_+(u)$ and $F_-(u)$ have been derived in the appendix.

In view of (25), (24) assumes the form

$$\frac{U_+(u)}{(u+\lambda_1)^{1/2}F_+(u)}=\frac{L_-(u)}{\dfrac{N}{iU_+(\alpha_1)}\dfrac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}} \tag{26}$$

where

$$U_+(u)=(u+\lambda_1)^{1/2}F_+(u). \tag{27}$$

So

$$L_-(u)=\frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}. \tag{28}$$

Hence the functions $A_1(u)$ and $A_2(u)$ are

$$A_1(u)=\frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}$$

and

$$A_2(u)=\frac{-N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\frac{\mu_1 s_1\gamma_1(\mu_1 s_1+\mu_2 s_2)}{\mu_2 s_2(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}. \tag{29}$$

The singular behaviour of the stress components for the scattered waves at the crack-tip is due to the divergence of the integrals around $x=y_j=0$ in (17). Making use of (29) and asymptotic expressions of the integrands of (17) for large values of u, we obtain near about the crack-tip,

$$(\tau_{xz})_1=\frac{B(1+i)}{s_1}\int_0^\infty u^{-1/2}\exp[-s_1 uY](\cos ux-\sin ux)\,du$$

$$(\tau_{xz})_2=\frac{-B(1+i)}{s_2}\int_0^\infty u^{-1/2}\exp[-s_2 u|Y|](\cos ux-\sin ux)\,du$$

$$(\tau_{yz})_1=-B(1+i)\int_0^\infty u^{-1/2}\exp[-s_1 uY](\cos ux+\sin ux)\,du$$

$$(\tau_{yz})_2=-B(1+i)\int_0^\infty u^{-1/2}\exp[-s_2 u|Y|](\cos ux+\sin ux)\,du \tag{30}$$

where

$$B=-\frac{N\mu_1 s_1}{2\pi(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)};\quad y_j=s_j\,Y\quad (j=1,2).$$

Using the results

$$\int_0^\infty u^{-1/2}\exp[-s_1uY]\cos ux\,\mathrm{d}x=(\pi/2)^{1/2}\left[\frac{(s_1^2y^2+x^2)^{1/2}+s_1y}{s_1^2y^2+x^2}\right]^{1/2}$$
$$\int_0^\infty u^{-1/2}\exp[-s_1uY]\sin ux\,\mathrm{d}x=(\pi/2)^{1/2}\left[\frac{(s_1^2y^2+x^2)^{1/2}-s_1y}{s_1^2y^2+x^2}\right]^{1/2}\tag{31}$$

the stresses near about the crack tip given by (30) can be evaluated. The displacement near about the crack tip can be obtained from the crack tip stresses by integration.

Now introducing the factor $\exp[i(M_1\lambda_1x-\omega t)]$ and taking the real part, the stresses and displacements near about the moving crack-tip are found to be equal to

$$\begin{bmatrix}(\tau_{yz})_j\\ \\ (\tau_{xz})_j\\ \\ v_j\end{bmatrix}=\mathrm{Re}\begin{bmatrix}K_1\left[\dfrac{(s_j^2\,Y^2+x^2)^{1/2}+x}{s_j^2\,Y^2+x^2}\right]^{1/2}\\ (-1)^j\dfrac{K_1}{s_1}\left[\dfrac{(s_j^2\,Y^2+x^2)^{1/2}-x}{s_j^2\,Y^2+x^2}\right]^{1/2}\\ (-1)^{j+1}\dfrac{2K_1}{\mu_js_j}[(x^2+s_j^2\,Y^2)^{1/2}-x]^{1/2}\end{bmatrix}\exp\left[i\left(M_1\lambda_1x-\omega t-\frac{\pi}{4}\right)\right]\tag{32}$$

where

$$K_1=(2/\pi)^{1/2}\frac{\mu_1\mu_2\Lambda_1\Lambda_2\,V_1\sin\Theta_1\sin\Theta_2}{(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)(\mu_1\Lambda_1\sin\Theta_1+\mu_2\Lambda_2\sin\Theta_2)}.\tag{33}$$

In the case of crack propagation in an isotropic elastic medium using the result $\mu_1=\mu_2$, $\rho_1=\rho_2$ and $F_+(\alpha_1)=1$, we obtain

$$K_1=(1/\pi)^{1/2}\mu_1\Lambda_1^{1/2}\,V_1(1-M_1)^{1/2}\sin(\Theta_1/2).\tag{34}$$

Putting $r=(x^2+y^2)^{1/2}$, $\tan\phi=|Y|/x$, the expression of displacements and stresses given by (32) near about the moving crack-tip is found to be equal to

$$v_1=\frac{2K_1}{\mu_1s_1}r^{1/2}\{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{3/2})$$

$$v_2=-\frac{2K_1}{\mu_2s_2}r^{1/2}\{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{3/2})$$

$$(\tau_{yz})_1=\frac{K_1}{r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{yz})_2 = \frac{K_1}{r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{xz})_1 = -\frac{K_1}{s_1}\frac{1}{r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2})$$

$$(\tau_{xz})_2 = \frac{K_1}{s_2}\frac{1}{r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+(\pi/4))+0(r^{1/2}). \tag{35}$$

Taking the value of K_1 given by (34), the results given by (35) agree with the results of the crack propagation in an isotropic elastic medium as given by Jahanshahi [3].

When the crack is stationary, the corresponding results of stresses and displacements near about the crack-tip can be derived by making M_1 and M_2 approach zero and are given by

$$(\tau_{yz})_1 = K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})$$

$$(\tau_{yz})_2 = K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})$$

$$(\tau_{xz})_1 = -K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2})$$

$$(\tau_{xz})_2 = K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{1/2}) \tag{36}$$

and

$$v_1 = \frac{2\sqrt{2}K_1^*}{\mu_1}(r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{3/2})$$

$$v_2 = \frac{-2\sqrt{2}K_1^*}{\mu_2}(r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\tfrac{1}{4}\pi)+0(r^{3/2}) \tag{37}$$

where

$$K_1^* = \sqrt{2/\pi}\,\frac{\mu_1\mu_2\Lambda_1\Lambda_2 V_1\sin\Theta_1\sin\Theta_2}{(\Lambda_1\cos\phi_1+\Lambda_1)^{1/2}F_+^*(\Lambda_1\cos\phi_1)(\mu_1\Lambda_1\sin\phi_1+\mu_2\Lambda_2\sin\phi_2)} \tag{38}$$

and

$$F_+^*(\Lambda_1\cos\phi_1) = \exp\left[\frac{1}{\pi}\int_{\Lambda_1}^{\Lambda_2}\tan^{-1}\left\{\frac{\mu_1(s^2-\Lambda_1^2)^{1/2}}{\mu_2(\Lambda_2^2-s^2)^{1/2}}\right\}\frac{ds}{s+\Lambda_1\cos\phi_1}\right]. \tag{39}$$

If we put $\mu_1=\mu_2$, $\rho_1=\rho_2$, the corresponding results of the stationary crack in an isotropic elastic medium are found to be

$$(\tau_{yz})_{1,2} = V_1(\sin\tfrac{1}{2}\Theta_1)(\cos\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2}+0(r^{1/2})$$

$$(\tau_{xz})_{1,2} = \mp V_1(\sin\tfrac{1}{2}\Theta_1)(\cos\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2}+0(r^{1/2})$$

and

$$v_{1,2} = \pm V_1(\sin\tfrac{1}{2}\Theta_1)(\sin\tfrac{1}{2}\phi)\cos(\Omega t+\tfrac{1}{4}\pi)\left[\frac{8\Lambda_1 r}{\pi}\right]^{1/2}+0(r^{3/2}) \tag{40}$$

which are same as given by Jahanshahi [3].

3. Results and discussion

K_1 given by (33) is the dynamic stress intensity factor at the moving crack-tip and K_1^* given by (38) is the value of the corresponding quantity when the crack is stationary. The variation of K_1/K_1^* with the values of V/C_2 where V is the crack speed has been depicted graphically for the following two sets of materials.

First set:

Wrought iron	$\rho_1 = 7{\cdot}8\,\mathrm{g/cm^3}$,	$\mu_1 = 7{\cdot}7 \times 10^{11}\,\mathrm{dyn/cm^2}$
Copper	$\rho_2 = 8{\cdot}96\,\mathrm{g/cm^3}$,	$\mu_2 = 4{\cdot}5 \times 10^{11}\,\mathrm{dyn/cm^2}$

Second set:

Steel	$\rho_1 = 7{\cdot}6\,\mathrm{g/cm^3}$,	$\mu_1 = 8{\cdot}32 \times 10^{11}\,\mathrm{dyn/cm^2}$
Aluminium	$\rho_2 = 2{\cdot}7\,\mathrm{g/cm^3}$,	$\mu_2 = 2{\cdot}63 \times 10^{11}\,\mathrm{dyn/cm^2}$.

It is found that in both the cases the stress intensity factor gradually decreases with an increase in the value of V/C_2 and approaches zero as $V/C_2 \to 1$; the decrease in the value of K_1/K_1^* for the second set being more rapid than for the first set. We also find that in both the cases for any fixed value of C_1/C_2, K_1/K_1^* decreases with decrease in the value of Θ.

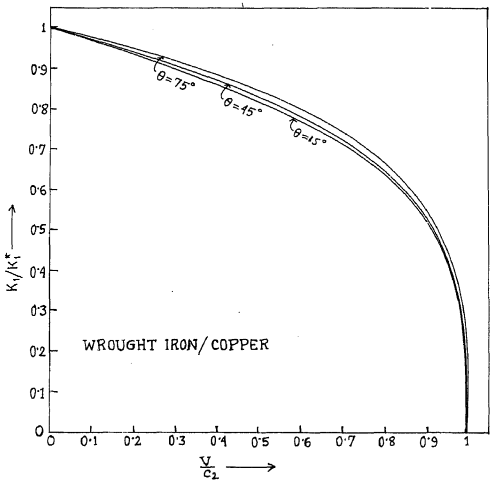

Figure 2. Stress intensity factor vs dimensionless crack speed (wrought iron/copper).

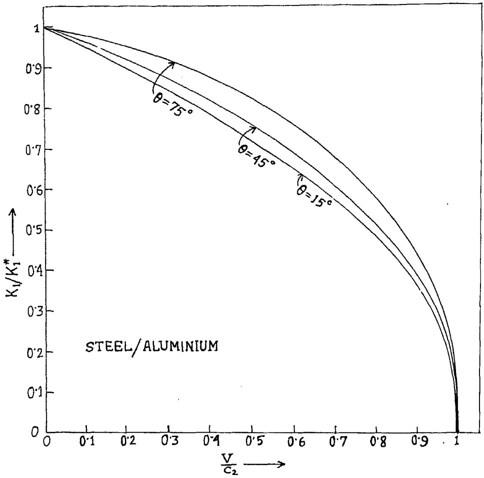

Figure 3. Stress intensity factor vs dimensionless crack speed (steel/aluminium).

Appendix

Factorization of $F(\xi)$ into $F_+(\xi)$ and $F_-(\xi)$

Consider

$$F(\xi) = \frac{(\mu_1 s_1 + \mu_2 s_2)\gamma_2}{(\mu_1 s_1 \gamma_1 + \mu_2 s_2 \gamma_2)} \tag{A1}$$

The branch points of $F(\xi)$ are at $\xi = \lambda_1, \; -\lambda_1, \; \lambda_2 + \beta_2, \; -(\lambda_2 - \beta_2)$ where

$$-(\lambda_2 - \beta_2) < -\lambda_1 < \lambda_1 < \lambda_2 + \beta_2 \text{ since } C_2 < C_1.$$

Since $F(\xi) \to 1$ as $|\xi| \to \infty$, $F(\xi)$ possesses no singularity within the rectangular contour (shown in figure 4), by Cauchy's residue theorem we can write

$$\log F(\xi) = \frac{1}{2\pi i} \int_{c+ + c-} \frac{\log F(s)}{s - \xi} \mathrm{d}s \tag{A2}$$

$$= \frac{1}{2\pi i} \int_{c+} \frac{\log F(s)}{s - \xi} \mathrm{d}s + \frac{1}{2\pi i} \int_{c-} \frac{\log F(s)}{s - \xi} \mathrm{d}s$$

$$\log F(\xi) = \log F_+(\xi) + \log F_-(\xi), \tag{A3}$$

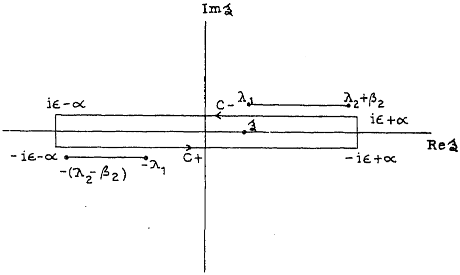

Figure 4. Rectangular contour in the complex ξ-plane.

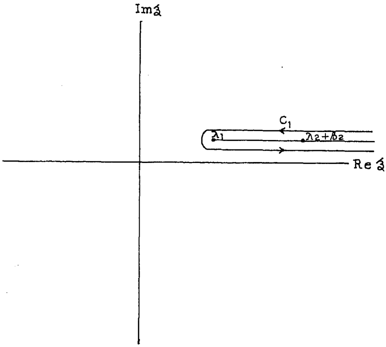

Figure 5. Path of integration C_1 round the branch cut.

where $F_+(\xi)$ and $F_-(\xi)$ are analytic in the upper and lower half of the complex ξ-plane respectively and can be expressed as

$$F_+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c+} \frac{\log F(s)}{s-\xi}\,ds\right]$$

and

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c-} \frac{\log F(s)}{s-\xi}\,ds\right]. \tag{A4}$$

In order to evaluate $F_-(\xi)$ the path of integration C_- can be deformed to the path C_1 round the branch cut through λ_1 and $\lambda_2+\beta_2$ as shown in figure 5.

After a little algebraic manipulation it can be shown that

$$F_{-}(\xi)=\exp\left[\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\log\left\{1+i\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}\mathrm{d}s\right.$$
$$\left.-\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\log\left\{1-i\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}\mathrm{d}s\right] \tag{A5}$$

which after simplification becomes

$$F_{-}(\xi)=\exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2+\beta_2}\frac{1}{s-\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\}\mathrm{d}s\right] \tag{A6}$$

where

$$m_1=\frac{\mu_1 s_1}{\mu_1 s_1+\mu_2 s_2}\quad\text{and}\quad m=\frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2}. \tag{A7}$$

Similarly it can be shown that

$$F_{+}(\xi)=\exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2-\beta_2}\frac{1}{s+\xi}\tan^{-1}\left\{\frac{m_1}{m_2}\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2+s)^2]^{1/2}}\right\}\mathrm{d}s\right]. \tag{A8}$$

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Continuous dependence for integrodifferential equations with infinite delay

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Abstract. Continuous dependence for integrodifferential equation with infinite delay

$$\dot{x} = h(t, x) + \int_{-\infty}^{t} q(t, s, x(s))\mathrm{d}s + F(t, x(t), Sx(t)) \ t \geqslant 0$$

$x(t) = \phi(t)$

where $Sx(t) = \int_{-\infty}^{t} k(t, s, x(s))\mathrm{d}s$ is studied under the assumption of existence of unique solution.

Keywords. Continuous dependence; infinite delay; integrodifferential equation.

1. Introduction

In this paper we consider the continuous dependence of solutions on initial functions for the integro-differential equation

$$\dot{x} = h(t, x) + \int_{-\infty}^{t} q(t, s, x(s))\mathrm{d}s + F(t, x(t), Sx(t)), \quad t \geqslant 0$$

$$x(t) = \phi(t), \quad -\infty < t \leqslant 0 \tag{1}$$

where $h \in C[J \times R^n, R^n]$, $q \in C[J \times J \times R^n, R^n]$, $F \in C[J \times R^n \times R^n, R^n]$, $Sx(t) = \int_{-\infty}^{t} k(t, s, x(s))\mathrm{d}s$ with $k \in C[J \times J \times R^n, R^n]$ and $J = [0, T]$. Continuous dependence of solutions to functional differential equations with infinite delay was discussed by Hale and Kato [3] and Hino [4]. The stability properties of (1) without delay was considered by Elaydi and Sivasundaram [2].

For a positive continuous nonincreasing function g defined on the interval $I = (-\infty, 0]$ with $g(0) = 1$, let X_g be the space of all continuous functions $\phi\colon I \to R^n$ for which

$$|\phi|_g = \sup\{|\phi(t)|/g(t) : t \in I\} < \infty,$$

where $|\cdot|$ denotes any norm in R^n. Then X_g is a Banach space with respect to the norm $|\cdot|_g$. This norm was introduced by Burton [1] to study qualitative theory for some functional differential equations.

The equivalent form of (1) can be written as

$$\dot{x}(t) = h(t,x) + \int_{-\infty}^{0} q(t,s,\phi(s))\,ds + \int_{0}^{t} q(t,s,x(s))\,ds$$

$$+ F(t,x(t)), \int_{-\infty}^{0} k(t,s,\phi(s))\,ds + \int_{0}^{t} k(t,s,x(s))\,ds, t \geqslant 0$$

$$x(t) = \phi(t) \text{ for } t \leqslant 0 \tag{2}$$

where $\phi \in X_g$ is an initial function.

2. Basic assumptions

For simplicity we put

$$Q(t,\phi) = \int_{-\infty}^{0} q(t,s,\phi(s))\,ds,$$

$K(t,\phi) = \int_{-\infty}^{0} k(t,s,\phi(s))ds$ and we assume the following hypotheses

(i) The initial value problem (1) has a unique solution $x(t,\phi)$ defined on J for each $\phi \in X_g$

(ii) The improper Riemann integrals

$$Q(t,\phi) = \lim_{c\to\infty} \int_{-c}^{0} q(t,s,\phi(s))\,ds$$

and

$$K(t,\phi) = \lim_{c\to\infty} \int_{-c}^{0} k(t,s,\phi(s))\,ds$$

exist and they are continuous in $t \in J$ for each $\phi \in X_g$.

(iii) For any $r > 0$ there exist positive integrable functions $m_r(t)$ and $n_r(t)$ on J such that $|Q(t,\eta)| \leqslant m_r(t)$ and $|K(t,\eta)| \leqslant n_r(t)$ if $t \in J$ and $\eta \in B(r)$. Here $B(r)$ is a closed ball $\{\eta \in X_g : |\eta| \leqslant r\}$, $r > 0$ and it is clear that $\eta \in B(r)$ if and only if $|\eta(t)| \leqslant rg(t)$ for all $t \in I$.

(iv) There exist integrable functions $a(t)$ and $b(t)$ such that $|F(t,x,Sx)| \leqslant a(t) + b(t)(|x| + |Sx|)$.

Lemma 1. [5] *Suppose $Q(t,\phi)$ and $K(t,\phi)$ exist and are continuous in $t \in J$ for each $\phi \in X_g$. If a sequence $\{\phi_n\} \subset X_g$ is bounded (with respect to the norm $|\cdot|_g$) and if it converges to $\phi \in X_g$ a.e on I, then $\{Q(t,\phi_n)\}$ converges to $Q(t,\phi)$ and $\{K(t,\phi_n)\}$ converges to $K(t,\phi)$ at every point $t \in J$.*

3. Continuous dependence of solutions

Theorem 1. *Suppose the hypotheses (i) to (iv) hold. If a bounded sequence $\{\phi_n\}$ in X_g converges to $\phi \in X_g$ a.e on I and if $\lim_{n\to\infty} \phi_n(0) = \phi(0)$, then $\{x(t,\phi_n)\}$ converges to $x(t,\phi)$ uniformly on J.*

Proof. Let us denote the solutions $x(t)=x(t,\phi)$ and $x_n(t)=x(t,\phi_n)$. Let $r>0$ be a bound for $\{\phi_n\}$ and ϕ, and let N and L be positive numbers satisfying $|x(t)|<N$ for $0\leqslant t\leqslant T$ and

$$L>N+\int_0^T(m_r(s)+a(s))\,\mathrm{d}s+\int_0^T b(s)n_r(s)\,\mathrm{d}s+(N+DT)\int_0^T b(s)\,\mathrm{d}s.$$

Furthermore, choose a $\tau\in(0,T]$ satisfying

$$\tau(H+CT)\leqslant L-N-\int_0^T(m_r(s)+a(s))\,\mathrm{d}s$$
$$-\int_0^T b(s)n_r(s)\,\mathrm{d}s-(N+DT)\int_0^T b(s)\,\mathrm{d}s,$$

where

$$H=\max\{|h(t,x)|:0\leqslant t\leqslant T,|x|\leqslant L\},$$
$$C=\max\{|q(t,s,x)|:0\leqslant s\leqslant t\leqslant T,|x|\leqslant L\},$$
$$D=\max\{|k(t,s,x)|:0\leqslant s\leqslant t\leqslant T,|x|\leqslant L\}.$$

Since $x_n(0)=\phi_n(0)\to\phi(0)=x(0)$ as $n\to\infty$, it follows that $|x_n(0)|<N$ for large n.

Now we shall show that $|x_n(t)|<L$ for $0\leqslant t\leqslant\tau$ if $|x_n(0)|<N$. Suppose the contrary, then there exists a $t_0\in[0,\tau]$ such that $|x_n(t_0)|=L$ and $|x_n(t)|<L$ for $0\leqslant t\leqslant t_0$.

By (iii), (iv) and (2) we obtain that

$$|\dot{x}_n(t)|\leqslant H+m_r(t)+CT+a(t)+b(t)(N+DT+n_r(t)),\quad\text{for } 0\leqslant t\leqslant t_0.$$

Therefore it follows that

$$\begin{aligned}|x_n(t)|&\leqslant x_n(0)+(H+CT)t+\int_0^t(m_r(s)+a(s))\,\mathrm{d}s\\&\quad+\int_0^t b(s)n_r(s)\mathrm{d}s+(N+DT)\int_0^t b(s)\mathrm{d}s\\&<N+(H+CT)\tau+\int_0^T(m_r(s)+a(s))\mathrm{d}s\\&\quad+\int_0^T b(s)n_r(s)\mathrm{d}s+(N+DT)\int_0^T b(s)\mathrm{d}s\\&\leqslant L.\end{aligned}$$

That is $|x_n(t)|<L$ which contradicts $|x_n(t_0)|=L$. Thus we get the assertion.

By using (iii), (iv) and (2) again, we obtain that

$$|\dot{x}_n(t)|\leqslant H+m_r(t)+CT+a(t)+b(t)(N+DT+n_r(t))$$

on $[0,\tau]$ for large n, and hence $\{x_n\}$ is equicontinuous there. Therefore $\{x_n\}$ includes a subsequence $\{x_{n_j}\}$ which is uniformly convergent on $[0,\tau]$. On the other hand x_n

satisfies,

$$x_n(t) = x_n(0) + \int_0^t h(s, x_n(s))\,ds + \int_0^t Q(s, \phi_n)\,ds + \int_0^t \int_0^s q(s, u, x_n(u))\,du\,ds$$
$$+ \int_0^t F(s, x_n(u)), K(s, \phi_n) + \int_0^t k(s, u, x_n(u))\,du\,ds$$

for $0 \leqslant t \leqslant \tau$. Hence it follows from Lemma (i), (iii) and (iv) that $\{x_{n_j}\}$ converges to a solution of (2). Since x is the unique solution of (2), the sequence $\{x_n\}$ itself converges to x uniformly on $[0, \tau]$.

When $\tau < T$, by the same argument as in the above, we can show that $\{x_n\}$ converges to x uniformly on $[\tau, 2\tau] \cap J$. Repeating this process, we arrive at the conclusion of the theorem.

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Curves on threefolds with trivial canonical bundle

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Abstract. C H Clemens has shown that homologically trivial codimension two cycles on a general hypersurface of degree five and dimension three form a subgroup of infinite rank inside the intermediate jacobian. We generalize this to other complete intersection threefolds with trivial canonical bundle.

Keywords. Algebraic cycles; algebraic threefolds; intermediate jacobian; rational curves.

1. Introduction

This paper is devoted to the study of rational curves on complex threefolds with trivial canonical bundle. Clemens ([5] and [4]) has asked if a simply connected threefold which has trivial canonical bundle always contains smooth rational curves. As pointed out by V Srinivas, the ètale quotient of a product of three elliptic curves constructed by Igusa [7] is an example of a threefold with trivial canonical bundle and vanishing first Betti number which contains no rational curves; thus the hypothesis of simple connectivity is necessary.

In an earlier paper [2] Clemens has shown the existence of *rigid* rational curves on the generic quintic hypersurface. Further, it is shown (*loc. cit.*) that these curves generate a subgroup of infinite rank inside the Griffiths group of the generic quintic.

These results naturally raise the question as to whether the phenomenon of rigidity of all rational curves and infinite generation of the Griffiths group occurs for all *generic* simply connected K-trivial threefolds. However, it was pointed out by C Schoen that if the Picard number is greater than one, rational curves are not in general rigid. Hence the class of varieties for which one can expect the results on quintics to generalize is that of simply-connected K-trivial threefolds with Picard number 1.

In this paper we study some special examples of such varieties—the complete intersections in $\mathbf{P}^5$. We prove results analogous to those of Clemens for these complete intersection threefolds.

The organization of the paper is as follows:

In §2 we give a summary of the results of Clemens [2] which allow one to prove infinite rank. The methods are completely general and ought to find applications in other dimensions as well.

In § 3 we give a general construction to which the results of § 2 can be applied. Here again the basic construction is for curves on a general hyperplane section of a del Pezzo fourfold and ought to be generalizable.

In § 4 we show that the methods of the previous two sections apply to the complete intersections in $\mathbf{P}^5$. We also summarize the arguments in the form of a theorem. It should be possible to refine these methods to prove the results for complete intersection subvarieties of Grassmanians and other homogeneous spaces. However, for other simply connected K-trivial threefolds, there does not appear to be a method available.

In an Appendix we prove a Bertini type result which is needed in § 2.

2. Summary of Clemens results

Let S be a smooth curve, $\pi:\mathscr{X}\to S$ be a projective family of threefolds with χ smooth, and π smooth except at $o\in\mathscr{X}$ which is an ordinary double point in the fibre X_0 over $o\in S$. Let $\tilde{X}_0\to X_0$ be the blow up of the singular point and E be the exceptional divisor for p; then we have $E\cong\mathbf{P}^1\times\mathbf{P}^1$. Let $t\in S$ be given by an inclusion of the function field of S in the complex numbers, henceforth we refer to such a t as a geometric generic point of S. Let X_t be the geometric generic fibre of π.

Lemma 1. *With notation as above, the following are equivalent*:

(i) *The action of monodromy on* $H^3(X_t,\mathbf{Z})$ *is non-trivial.*
(ii) *The vanishing cycle* $\ker(H_3(X_t,\mathbf{Z})\to H_3(X_0,\mathbf{Z}))$ *is non-zero.*
(iii) *The Hodge structure* $H^3(X_0)$ *is* not *pure.*
(iv) *The morphism* $\mathrm{Pic}(\tilde{X}_0)\otimes\mathbf{Q}\to\mathrm{Pic}(E)\otimes\mathbf{Q}$ *is not surjective.*

Proof. Let $\delta\in H^3(X_t,\mathbf{Z})$ denote the "co-vanishing" cohomology class. The action of monodromy on $H^3(X_t,\mathbf{Z})$ is given by $x\mapsto x+(x,\delta)\delta$. Thus δ is trivial if and only if the monodromy action is trivial. This gives the equivalence of (i) and (ii).

In the following exact sequence of mixed Hodge structures

$$0\to H^3(X_0)\to H^3(X_t)_{\lim}\to\mathbf{Z}(-2),$$

the latter map is given by $x\mapsto(x,\delta)\delta$. Note that $H^3(X_t)_{\lim}$ is self-dual up to a twist and so $H^3(X_0)$ contains a $\mathbf{Z}(-1)$ if and only if δ is non-zero. In other words, the purity of $H^3(X_0)$ is equivalent to the triviality of δ. Thus we have the equivalence of (ii) and (iii).

Finally, we have the exact sequence of mixed Hodge structures

$$H^2(\tilde{X}_0)\to H^2(E)\to H^3(X_0)\to H^3(\tilde{X}_0),$$

which shows the $H^3(X_0)$ is not pure if and only if (iv) holds. □

Assume that one of the above equivalent conditions holds. Let $d:T\to S$ be a double cover ramified at $o\in T$ lying over $o\in S$. The normalization of $\mathscr{X}\times_S T$ has an ordinary double point; let $\mathscr{Y}$ be the result of blowing up this ordinary double point. The special fibre of $\mathscr{Y}\to T$ is the union of $\tilde{X}_0$ and a smooth quadric threefold Q which meet transversally along E.

In this situation Clemens [3] has constructed a Néron family $\mathscr{T} \to T$ of intermediate Jacobians over T such that the special fibre J_0 has two components $J_0^{\pm}$. Here

$$J_0^+ = H^3(X_0, C)/(F^2H^3(X_0, \mathbf{C}) + H^3(X_0, \mathbf{Z})),$$

an extension of a compact torus by $\mathbf{G}_m$. Further, in the diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & \mathbf{Z}(-1) & \to & H^3(X_0) & \to & H^3(\tilde{X}_0) & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & H^3(Q,E) & \to & H^3(\tilde{X}_0 \cup Q) & \to & H^3(\tilde{X}_0) & \to 0 \end{array}$$

the vertical maps are isomorphisms. Thus, this $\mathbf{G}_m$ can be identified with $\mathrm{Ext}^1_{\mathrm{MHS}}$ $(\mathbf{Z}(-2), H^3(Q, E))$.

Choose $p \in E$ and let C be the quadric cone in Q with vertex p. The intersection $C \cap E$ is a pair of lines meeting in p. We have an isomorphism

$$H^1(C-p, C \cap E - p)\cdot(-1) \cong H^3(Q, E).$$

Any pair of distinct lines L_1, L_2 which are distinct from the lines in $C \cap E$ give a non-trivial extension

$$0 \to H^1(C-p, C\cap E - p) \to H^1(C - L_1 - L_2, C \cap E - p)$$
$$\to \mathbf{Z}(-1)[L_1 - L_2] \to 0$$

and hence a non-trivial point of $\mathbf{G}_m$.

Let $\iota\colon T \to T$ be the involution corresponding to the double cover $d\colon T \to S$. This gives us $\iota\colon \mathscr{Y} \to \mathscr{Y}$ as well. The action on the special fibre of $\mathscr{Y} \to T$ is described as follows: ι acts trivially on $\tilde{X}_0$ and on Q it acts as the unique involution which fixes E. Further, we can lift ι to an action $\tilde{\iota}$ on $\mathscr{T}$. The action of $\tilde{\iota}$ on the special fibre is trivial on the identity component J_0^+ and is non-trivial on the remaining component J_0^-.

Lemma 2. *In the situation of Lemma 1 assume that the action of monodromy is non-trivial. Suppose in addition that we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{C} & \hookrightarrow & \mathscr{X} \\ p\downarrow & & \downarrow p \\ T & \xrightarrow{d} & S \end{array}$$

where d is the double cover as above and $\mathscr{C} \to T$ is a smooth family of connected curves which embeds into $\mathscr{X}$ in such a way that $o \in X_0$ lies on the special fibre C_0.

For each $t \in T$ different from o, the difference $\sigma(t) = C_t - C_{\iota(t)}$ gives a point in the intermediate Jacobian of $X_{d(t)}$. This extends to a section $\sigma\colon T \to \mathscr{T}$ such that $\sigma(o)$ is a non-trivial two-torsion class in the identity component J_0^+ of the special fibre.

Proof. The surface $\mathscr{C} \subset \chi$ is smooth and meets X_0 in C_0 with multiplicity two. Hence, we get two maps $m\colon \mathscr{C} \to \mathscr{X} \times T$ and $\iota(m)\colon \mathscr{C} \to \mathscr{X} \times T$ where the images meet along C_0 with multiplicity one. Let $\tilde{\mathscr{C}}$ be the blow up of $\mathscr{C}$ at $o \in \mathscr{C} \subset \mathscr{X}$ and let D denote the exceptional divisor of $\tilde{\mathscr{C}} \to \mathscr{C}$; then m and $\iota(m)$ give us two maps

$n: \mathscr{C} \to y$ and $\imath(n): \tilde{\mathscr{C}} \to y$, such that $n(D)$ and $\imath(n)(D)$ are distinct lines L_1, L_2 in Q that meet E in the same point p. Their difference then gives us a non-trivial class α in the $\mathbf{G}_m$ part of J_0^+.

For each $t \in T$ different from o, we see by continuity that the curves C_t and $C_{\imath(t)}$ are homologically equivalent in the fibre $X_{d(t)}$. Thus we have a class $\sigma(t) = C_t - C_{\imath(t)}$ which is homologically trivial in the Chow group of $X_{d(t)}$. The limiting class is

$$\sigma(o) = n(D) - \imath(n)(D) \in \mathrm{CH}^2(\tilde{X}_0 \cup Q),$$

for a suitable definition of the latter Chow group. Furthermore, $\sigma(t)$ gives a point in the intermediate Jacobian of the fibre $X_{d(t)}$ which extends to a section $\sigma: T \to \mathscr{T}$. Clearly $\sigma(o) = \alpha$ which is non-trivial. Now $\sigma(o) \in J_0^+$ is fixed by $\imath$; on the other hand from the expression above $\imath(\sigma(o)) = -\sigma(o)$, hence it is a non-trivial two torsion class in the identity component J_0^+. □

Let X be a smooth projective threefold with $H^4(X, \mathbf{Z}) = \mathbf{Z}$, and $\{C_d \subset X\}$ be an infinite collection of curves. For any codimension 2 linear section $C_0 \subset X$ we get classes

$$e_d = C_d - \frac{\deg C_d}{\deg C_0} C_0 \in J(X)$$

where $J(X)$ is the intermediate Jacobian of X. As in Clemens [2] we now give a sufficient condition for these classes to generate a subgroup of infinite ranks in $J(X)$.

Assume that (X, C_d) is the pair corresponding to the geometric generic point of S_d in a situation

$$\begin{array}{ccc} \mathscr{C}_d & \hookrightarrow & \mathscr{X}_d \\ p\downarrow & & \downarrow p \\ T^d & \xrightarrow{d} & S_d \end{array}$$

as in Lemma 2; here we have used the subscript to indicate dependence on d. Further assume that for each $l \neq d$ we have a commutative diagram which specializes to (X, C_l) at the geometric generic point of S,

$$\begin{array}{ccc} \mathscr{C}_{l,d} & \hookrightarrow & \mathscr{X}_d \\ \pi\downarrow & & \downarrow\pi \\ S_d & \cong & S_d \end{array}$$

where $\mathscr{C}_l \to S$ is a smooth family of connected curves with embeds into $\mathscr{X}$ in such a way that it misses the ordinary double point of X_0.

All the above data is defined for all d over a countably generated field over $\mathbf{Q}$. Hence it makes sense to assume that there is a geometric generic point $s_d \in S_d - o$ for each d, where the above data specializes to $(X, \{C_l\})$.

Lemma 3. *If* $(X, \{C_l\})$ *are as above then classes* e_l *generate a subgroup of infinite rank in the intermediate Jacobian* $J(X)$ *of* X.

Proof. As in Lemma 2 the action of $\imath$ fixes the class of C_0 since we have assumed

that $H^4(X, \mathbf{Z}) = \mathbf{Z}$. Thus we have an additional class

$$\imath(e_d) = e'_d = \imath(C_d) - \frac{\deg C_{d,\imath(t)}}{\deg C_0} C_0$$

in the intermediate Jacobian of X.

With notation as in the proof of Lemma 2, we have $e_d - e'_d = \sigma(t)$. The action of $\tilde{\imath}$ on the classes e_l for $l \neq d$ is trivial since the class of C_0 is fixed and C_l is fixed. Suppose that we have a relation $\Sigma_{\text{finite}} n_d e_d = 0$ then by applying $\tilde{\imath}$ to this relation we get $n_d e'_d + \Sigma_{l \neq d} n_l e_l = 0$. Thus we see that $n_d \sigma(t) = 0$. Then by degeneration we have $n_d \sigma(0) = 0$ and by Lemma 2 we see that n_d must be even.

For any relation $\Sigma_{\text{finite}} n_d e_d = 0$ in the intermediate Jacobian of X, this shows that n_d is even for all d. Thus e_d are independent mod 2. Now let G be the group generated by e_d. We can apply the following easy lemma to G to show that its rank is infinite. □

Lemma 4. *If G is an abelian group such that its torsion subgroup G_{tors} is a subgroup of $(\mathbf{Q}/\mathbf{Z})^r$, then we have*

$$\operatorname{rank}_{\mathbf{Q}}(G \otimes \mathbf{Q}) + r \geqslant \operatorname{rank}_{\mathbf{Z}/2\mathbf{Z}}(G \otimes \mathbf{Z}/2\mathbf{Z}).$$

Using Lemma 1 we can characterize the families $\mathscr{X}_d \to S_d$ by means of conditions on the special fibres $X_0 = X_{0,d}$. A precise meaning will be given to the deformation schemes in the examples considered in § 4.

1. X_0 has at most ordinary double points as singularities.
2. For $l \neq d$, the curves C_l are smooth and lie in the smooth locus of X_0 and the morphism $\mathbf{Def}(X_0, C_l) \to \mathbf{Def}(X_0)$ from the space of deformations of the pair (X_0, C_l) to the space of deformations of X_0 is étale at the point corresponding to (X_0, C_l).
3. C_d is a smooth curve in X_0 passing through *exactly one* ordinary double point $p \in X_0$ and the morphism $\mathbf{Def}(X_0, C_d) \to \mathbf{Def}(X_0)$ is doubly ramified along a divisor containing the point corresponding to (X_0, C_d).
4. Let $\tilde{X}_0$ be the blow-up of X_0 at all its ordinary double points, and let $\{E_q\}_{q \in (X_0)_{\text{sing}}}$ denote the exceptional divisors. If $p \in C_d$ is the special point then the image of

$$\operatorname{Pic}(\tilde{X}_0) \otimes \mathbf{Q} \to \otimes_{q \in (X_0)_{\text{sing}}} \operatorname{Pic}(E_q) \otimes \mathbf{Q}$$

does not contain $\operatorname{Pic}(E_p)$.

The next section will give a general procedure for constructing examples of such degenerations.

3. The principal construction

Let Y be a smooth del Pezzo fourfold, i.e. $Y \subset \mathbf{P}^n$ and $K_Y^{-1} \cong \mathcal{O}_Y \otimes \mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{O}_Y(1)$. Let $S \subset Y$ be a smooth surface such that S is the scheme theoretic intersection of Y with a linear subspace, i.e. if $V = \Gamma(Y, I_{S/Y} \otimes \mathcal{O}_Y(1))$, then we have a surjection

$$V \otimes \mathcal{O}_Y \twoheadrightarrow I_{S/Y} \otimes \mathcal{O}_Y(1) = I_{S/Y}(1).$$

Let $E \subset S$ be an exceptional divisor of the first kind, i.e. $E \cong \mathbf{P}^1$ and $N_{E/S} \cong \mathcal{O}_{\mathbf{P}^1}(-1)$.

Lemma 5. *Let Y, S and E be as above. We can find a hyperplane section X of Y containing S and smooth along E.*

In this situation, if **Hilb** $((X, E); Y)$ *denotes the space of deformations of the pair (X, E) in Y and* **Hilb** $(X; Y)$ *the deformation of X in Y then the natural morphism*

$$\mathbf{Hilb}\,((X, E); Y) \to \mathbf{Hilb}\,(X; Y)$$

is étale at the point corresponding to (X, E).

Proof. $N^*_{S/Y}(1)$ is generated by its global sections, in fact we have a surjection $V \otimes \mathcal{O}_S \twoheadrightarrow N^*_{S/Y}(1)$. Thus on restricting this to E we have

$$V \otimes \mathcal{O}_E \twoheadrightarrow N^*_{S/Y}(1)|_E.$$

Hence we can find a section $v \in V$ such that this gives a nowhere vanishing section of $N^*_{S/Y}(1)|_E$. Let X_v be the corresponding hyperplane section of Y. Then X_v contains S and is smooth along E.

Now we have an exact sequence of vector bundles on E.

$$0 \to N_{E/S} \to N_{E/X_v} \to N_{S/X_v}|_E \to 0.$$

Furthermore $K_{X_v} = K_Y \otimes \mathcal{O}_Y(X_v) \otimes \mathcal{O}_{X_v} \cong \theta_{x_v}$thus $\det N_{E/X_v} = K_E = \mathcal{O}_{\mathbf{P}^1}(-2)$, and so

$$N_{S/X_v}|_E = \det N_{E/X_v} \otimes N_{E/S}^{-1} = \mathcal{O}_{\mathbf{P}^1}(-1).$$

As a result the above sequence splits and $N_{E/X_v} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$.

The infinitesimal deformations of the pair (X_v, E) in Y are given by

$$U = \ker(H^0(E, N_{E/Y}) \oplus H^0(X, N_{X_v/Y}) \to H^0(E, N_{X_v/Y}|E)).$$

From the exact sequence

$$0 \to N_{E/X_v} \to N_{E/Y} \to N_{X_v/Y}|_E \to 0$$

we see that $H^0(E, N_{E/Y}) \cong H^0(E, N_{X_v/Y}|E)$ which yields the isomorphism $U \cong H^0 \cdot (X, N_{X_v/Y})$ under the natural morphism. $\square$

Let G be the vector bundle on S defined by the sequence

$$0 \to G^* \to V \otimes \mathcal{O}_S \to N^*_{S/Y}(1) \to 0.$$

Let f: $\mathbf{P}_S(G) \to \mathbf{P}(V^*)$ denote the natural map. For any point $v \in \mathbf{P}(V^*)$ such that f is étale over v, the set $f^{-1}(v)$ consists of finitely many points. The projections of these points give the singularities of X_v along S. It is easily seen (see Appendix A) that these are ordinary double points. We now need to choose v so that exactly one of these singularities lies on E and also ensure the rigidity of E in X_v for this choice of v. The first step is

Lemma 6. *Let N be a vector bundle on a smooth projective curve E, $V \subset \Gamma(E, N)$ be a*

space of sections such that $g:\mathbf{P}_E(N)\to\mathbf{P}(V)$ *is an embedding. Let* G *be the vector bundle defined by the exact sequence*

$$0\to G^*\to V\otimes\mathcal{O}_E\to N\to 0.$$

Then the map $f:\mathbf{P}_E(G)\to\mathbf{P}(V^*)$ *is birational to its image and the ramification locus of this morphism has codimension two in* $\mathbf{P}_E(G)$.

Proof. Now $f^*(\mathcal{O}_{\mathbf{P}(V^*)}(1))=\mathcal{O}_G(1)$ is the tautological line bundle on $\mathbf{P}_E(G)$ which is a quotient of π_1^*G where $\pi_1:\mathbf{P}_E(G)\to E$ is the natural projection. Observe the following diagram of Euler sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_G & = & \mathcal{O}_G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0\to & & \pi_1^*G^*\otimes\mathcal{O}_G(1) & \to & V\otimes\mathcal{O}_G(1) & \to & \pi_1^*N\otimes\mathcal{O}_G(1) & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0\to & & T_{\mathbf{P}_E(G)/E} & \to & f^*T_{\mathbf{P}(V^*)} & \to & \mathcal{K} & \to 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and the following diagram of exact sequences

$$\begin{array}{cccccccc}
0\to & T_{\mathbf{P}_E(G)/E} & \to & T_{\mathbf{P}_E(G)} & \to & \pi_1^*T_E & \to 0 \\
 & \| & & \downarrow & & \downarrow & \\
0\to & T_{\mathbf{P}_E(G)/E} & \to & f^*T_{\mathbf{P}(V^*)} & \to & \mathcal{K} & \to 0.
\end{array}$$

These show us that df is computed by a map on $\mathbf{P}_E(G)$

$$\phi:\pi_1^*T_E\to\pi_1^*N\otimes\mathcal{O}_G(1).$$

Similarly, if $\pi_2:\mathbf{P}_E(N)\to E$ denotes the natural projection, one shows that dg is computed by a map on $\mathbf{P}_E(N)$

$$\gamma:\pi_2^*T_E\to\pi^*G\otimes\mathcal{O}_N(1).$$

In fact, if $\pi:\mathbf{P}_E(G)\times_E\mathbf{P}_E(N)\to E$ is the fibre product we have a natural morphism

$$\psi:\pi^*(T_E)\to\mathcal{O}_G(1)\otimes\mathcal{O}_N(1)$$

such that $\phi=(p_1)_*(\psi)$ and $\gamma=(p_2)_*(\psi)$.

We are given that g is an embedding, and thus dg and also γ are inclusions of vector bundles. This gives us a subvariety

$$D=\mathbf{P}_{\mathbf{P}_E(N)}(\operatorname{coker}\gamma)\subset\mathbf{P}_E(G)\times_E\mathbf{P}_E(N)$$

which is precisely the vanishing locus of ψ. It follows that $D\subset\mathbf{P}(V^*)\times\mathbf{P}_E(N)$ is precisely the collection of pairs (v,n), such that the hyperplane section of $\mathbf{P}_E(N)$ defined by v is singular at n. Let $D'\subset\mathbf{P}(V^*)$ be the image of D; this is the dual variety of

$\mathbf{P}_E(N) \to E$ and the divisor $P_E(\mathscr{T})$, for the quotient $N \twoheadrightarrow (N/v \cdot \mathcal{O}_E)/\text{torsion} \cong \mathscr{T}$ with $(v, n), v \in D'$ a smooth point and $n \in \mathbf{P}(V)$ such that the hyperplane in $\mathbf{P}(V^*)$ defined by n is tangent to D' at v. The fibre of the map $D \to D'$ is a projective space at the general point of D'.

If $v \in \mathbf{P}(V^*)$ is in the image $f(\mathbf{P}_E(G))$, then the corresponding hyperplane section of $\mathbf{P}_E(N)$ contains a fibre of π. Thus this hyperplane section is singular. Furthermore, for any $v \in \mathbf{P}(V^*)$, the hyperplane section is the union of finitely many fibres of $\mathbf{P}_E(N) \to E$ and the divisor $P_E(\mathscr{T})$, for the quotient $N \twoheadrightarrow (N/v \cdot \mathcal{O}_E)/\text{torsion} \cong \mathscr{T}$ with rank 1 kernel. Thus, for any $v \in \mathbf{P}(V^*)$ the singularities of the corresponding hyperplane section are contained in finitely many fibres of $\mathbf{P}_E(N) \to E$. In particular, D' is the image of $\mathbf{P}_E(G)$ and the map $\mathbf{P}_E(G) \to D'$ is generically finite. Combined with the fact that $D \to D'$ has general fibre a projective space we see that D, $\mathbf{P}_E(G)$ and D' are birational.

The cokernel of ϕ is supported on the subset of $\mathbf{P}_E(G_E)$ where the birational map $D \to \mathbf{P}_E(G_E)$ has at least 1-dimensional fibres. Since D is irreducible this is of codimension $\geqslant 2$ in $\mathbf{P}_E(G_E)$. □

Let X be a hyperplane section of Y which contains S and has exactly one ordinary double point lying on E and no other singularities on E; such an X will be provided using the above lemma. We must find a condition for

$$\mathbf{Hilb}((X, E); Y) \to \mathbf{Hilb}(X; Y)$$

to be ramified to order two along a divisor containing (X, E).

Let $\varepsilon: \tilde{Y} \to Y$ be the blow up of Y along S; the exceptional divisor is $P = \mathbf{P}_S(N^*_{S/Y}(1))$; we have a ruled surface $Q = \mathbf{P}_E(N^*_{S/Y}(1)|_E)$ contained in P. Let $\tilde{X}$ be the strict transform of X in $\tilde{Y}$. Then $\tilde{X}$ meets P in a smooth surface $\tilde{S}$ which is the blow up of S at the finitely many ordinary double points of X; one of these is a point $e \in E$. Let F_e be the exceptional divisor of $\tilde{S} \to S$ over e and $\tilde{E}$ be the strict transform of E in $\tilde{S}$. Then $\tilde{X}$ meets Q in $\tilde{E} \cup F_e$; note that $\tilde{E}$ is a section of $Q \to E$ and F_e is the fibre of $Q \to E$ over e. Further, $\tilde{X}$ is smooth along $\tilde{E}$, thus we see that there is a natural map $N_{\tilde{E}/\tilde{X}} \to N_{Q/\tilde{Y}}|_E$, the cokernel of which can be canonically identified with the fibre of $N_{\tilde{X}/\tilde{Y}}$ at $\tilde{e} = \tilde{E} \cap F_e$. We have a diagram of exact sequences,

$$\begin{array}{ccccccccc}
0 & \to & N_{\tilde{E}/\tilde{S}} & \to & N_{\tilde{E}/\tilde{X}} & \to & N_{\tilde{S}/\tilde{X}}|_{\tilde{E}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N_{Q/P} & \to & N_{Q/\tilde{Y}}|_{\tilde{E}} & \to & N_{P/\tilde{Y}}|_{\tilde{E}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N_{E/S} & \to & N_{E/Y} & \to & N_{S/Y}|_E & \to & 0
\end{array}$$

where $N_{P/\tilde{Y}}|_{\tilde{E}}$ can be computed to be $\mathcal{O}_E$ and the inclusion $N_{P/\tilde{Y}}|_{\tilde{E}} \subsetneq N_{S/Y}|_E$ is the one induced by the given morphism $E \cong \tilde{E} \subset Q$.

The last row of the diagram gives us $\Gamma(E, N_{E/Y}) \cong \Gamma(E, N_{S/Y}|_E)$ and thus we have a lift $N_{P/\tilde{Y}}|_{\tilde{E}} \to N_{E/Y}$, in fact it is easily shown that the section actually lifts to $N_{Q/\tilde{Y}}|_{\tilde{E}}$ to split the middle sequence. In order to show that $N_{\tilde{E}/\tilde{X}}$ has no sections we must show that the image of this splitting maps non-trivially under the morphism $N_{Q/\tilde{Y},\tilde{e}} \to N_{\tilde{X}/\tilde{Y},\tilde{e}}$. We shall show this by varying the choice of $\tilde{X}$.

Lemma 7. Let $f: \mathbf{P}_S(G) \to \mathbf{P}(V^*)$ be the natural map. Assume that $N = \mathbf{P}_E(G|_E) \subset \mathbf{P}_S(G)$ is not entirely contained in the ramification locus of f. Further assume that $f|_N$ is birational to its image and is unramified outside codimension 2.

Then, there is a hyperplane section X of Y containing S which has at most ordinary double points as singularities. Further, exactly one of these singularities lies on E and **Hilb** *$((X, E); Y) \to$* **Hilb***$(X; Y)$ is ramified to order two along a divisor containing the point (X, E).*

Proof. Let $\Gamma_\pi \subset N \times E$ denote the graph of $\pi: N \to E$. On $N \times E$ we have a map

$$\Psi: \mathcal{O}_{N \times E}(\Gamma_\pi) \to p_1^* \mathcal{O}_G(1) \otimes p_2^*(N_{S/Y}^*(1)|_E)$$

which restricts to ψ on $\Gamma_\pi \cong N$. Using the isomorphisms

$$\mathcal{O}_{N \times E}(\Gamma_\pi) \cong p_1^* \pi^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(1) \text{ and } N_{S/Y}^*(1)|_E \cong N_{S/Y} \otimes \mathcal{O}_{\mathbf{P}^1}(1)$$

this is equivalent to a map

$$p_1^*(\mathcal{L}) = p_1^*(\mathcal{O}_N(-1) \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(1)) \to p_2^*(N_{S/Y}|_E).$$

Let Z be the vanishing locus for Ψ. Then Z meets Γ_π in the vanishing locus for ψ which is given to be of codimension 2. Further, Γ_π meets every effective divisor in $N \times E$ and thus Z is also of codimension 2 in $N \times E$; in particular, the map $\mathcal{L} \hookrightarrow p_2^*(N_{S/Y}|_E)$ is saturated. Hence, if $U = N \times E - Z$, we have a morphism $p: U \to Q = \mathbf{P}_E(N_{S/Y}^*(1)|_E)$ such that $p^* \mathcal{O}_Q(1) = \mathcal{L}^{-1} \otimes p_2^*(\mathcal{O}_Y(1)|_E)$.

The sequence $0 \to N_{Q/P} \to N_{Q/\tilde{Y}} \to N_{P/\tilde{Y}}|_Q \to 0$ on Q pulls back under p to

$$0 \to p_2^* N_{E/S} \to p^* N_{Q/\tilde{Y}} \to p_1^*(\mathcal{L}) \to 0$$

since $N_{P/\tilde{Y}}|_Q \cong \mathcal{O}_Q(1) \otimes \pi^*(\mathcal{O}_Y(-1)|_E)$. Taking direct images under p_1 we see that this sequence splits to give a map $p_1^*(\mathcal{L}) \to p^* N_{Q/\tilde{Y}}$. As seen in the arguments preceding the lemma we have a natural surjection on Γ_π from the restriction of $N_{Q/\tilde{Y}}$ to the restriction of the pull back $p^* \mathcal{O}_{\tilde{Y}}(\tilde{X})$. Note that $\mathcal{O}_{\tilde{Y}}(\tilde{X}) \cong \mathcal{O}_{\tilde{Y}}(-P) \otimes \varepsilon^* \mathcal{O}_Y(1)$ which restricts on Q to $\mathcal{O}_Q(1)$. In order to show that we have rigidity for $\tilde{E}$ in $\tilde{X}$ we need to show that the composite morphism on $\Gamma_\pi \cong N$

$$\mathcal{L} \to p^* N_{Q/\tilde{Y}}|_{\Gamma_\pi} \to p^* \mathcal{O}_Q(1)|_{\Gamma_\pi}$$

is non-zero. The kernel of the second morphism can be computed to be

$$(\pi^* N_{E/S} \otimes \mathcal{L}) \otimes (\mathcal{L}^{-1} \otimes \pi^*(\mathcal{O}_Y(1)|_E))^{-1} \cong \mathcal{L}^{\otimes 2} \otimes \pi^*(\mathcal{O}_Y(1)|_E \otimes N_{E/S}).$$

So, if $\mathcal{L}$ landed completely inside this on U, we would have a non-trivial section of $\mathcal{L} \otimes \pi^*(\mathcal{O}_Y(-1)|_E \otimes N_{E/S})$ which is isomorphic to $\mathcal{O}_N(-1) \otimes \pi^*(\mathcal{O}_Y(-1)|_E)$. Since the complement of U is of codimension two any such section would extend to all of N and that is clearly impossible.

To summarize, at some suitably chosen point of n we have

1. If $v = f(n) \in \mathbf{P}(V^*)$ then X_v has no singularities other than finitely many ordinary double points on S.

2. There is exactly one ordinary double point of X_v which lies on E.
3. The curve $\tilde{E} \subset \tilde{X}$ is rigid; in fact, from the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1}(-2) \to N_{\tilde{E}/\tilde{X}} \to \mathcal{O}_{\mathbf{P}^1} \to 0$$

and the fact that $\Gamma(E, N_{\tilde{E}/\tilde{X}}) = 0$, we see that $N_{\tilde{E}/\tilde{X}} \cong \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$.

The curve $\tilde{E} \cup F_e$ is an exceptional tree of curves of the first kind on $\tilde{S}$ and thus by an argument similar to the one in Lemma 5 it deforms into nearby $\tilde{X}$'s. Thus each of the curves $\tilde{E}$ and $\tilde{E} \cup F_e$ deforms to nearby $\tilde{X}$'s. This gives the result. □

Now, assume that $V = \Gamma(Y, I_{S/Y}(1)) \cong \Gamma(\tilde{Y}, \mathcal{O}_Y(1) \otimes \mathcal{O}_{\tilde{Y}}(-P))$ gives a very ample linear system on $\tilde{Y}$. Then V is also very ample on $Q = \mathbf{P}_E(N^*_{S/Y}(1)|_E)$ so that we can apply Lemma 6. Furthermore, for a general $v \in V$, if $\tilde{X}_v$ denotes the corresponding hyperplane section of $\tilde{Y}$, then we have $\mathrm{Pic}(X_v) \cong \mathrm{Pic}(\tilde{Y}) = \mathrm{Pic}(Y) \oplus \mathbf{Z}[P]$. Now, the blow up of the singularities of X_v gives the same result as blowing up all the fibres of $P \to S$ which lie in $\tilde{X}_v$ from this we see that the hypothesis (4) at the end of § 2 is satisfied.

Finally, assume that S has infinitely many exceptional curves of the first kind $\{E_d\}$. We can then choose an infinite subcollection $\{C_d\}$ so that the images of $\mathbf{P}_{C_d}(G|_{C_d})$ in $\mathbf{P}(V^*)$ are distinct. Then by the above lemmas and subsequent discussion, it is possible to choose, for each d a point v_d so that the hyperplane section X_{v_d} satisfies the conditions stated at the end of § 2.

To summarize the hypothesis on S and Y:

1. $Y \subset \mathbf{P}^n$ is such that $K_Y^{-1} \cong \mathcal{O}_Y \otimes \mathcal{O}_{\mathbf{P}^n}(1)$
2. $S \subset Y$ is such that if $V = \Gamma(Y, I_{S/Y}(1))$ then, V generates $I_{S/Y}(1)$ at stalks
3. If $\tilde{Y}$ is the blow up of Y along S, then the map $\tilde{Y} \to \mathbf{P}(V)$ is an embedding
4. S contains infinitely many exceptional curves of the first kind

We shall produce such examples in the next section.

4. Examples

Let $b: S \to \mathbf{P}^2$ be the surface obtained by blowing up 9 points in general position. Let $H = b^* \mathcal{O}_{\mathbf{P}^2}(1)$ and let E_i denote the exceptional curves in S over the points in $\mathbf{P}^2$ which have been blown up. Let C be the unique elliptic curve in the linear system $|3H - \Sigma_{i=1}^9 E_i|$.

Lemma 8. *With notation as above, S can be embedded in $\mathbf{P}^5$ by the linear system $|4H - \Sigma_{i=1}^9 E_i|$. Further, we have a surjection*

$$\mathcal{O}_{\mathbf{P}^5}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbf{P}^5}(-3) \to I_{S/\mathbf{P}^5}.$$

Proof. We have a short exact sequence on S

$$0 \to \mathcal{O}_S(H) \to \mathcal{O}_S\left(4H - \sum_{i=1}^{9} E_i\right) \to \mathcal{O}_C\left(4H - \sum_{i=1}^{9} E_i\right) \to 0.$$

Since $H^1(S,\mathcal{O}_S(H))=H^1(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(1))=0$, the associated long exact sequence on cohomology shows that the linear system $D=|4H-\Sigma_{i=1}^9 E_i|$ on S is of dimension five and has no base points on C. In addition, the linear system D contains all curves of the form $C+H$ where H is the pull-back of a line in $\mathbf{P}^2$, so it has no base points outside C. Hence we have a base-point free linear system on S and a map to $\mathbf{P}^5$. We can use the results of Nagata [8] to show that the linear system D in fact embeds S in $\mathbf{P}^5$ as a surface of degree 7.

Let A be a general curve in the linear system D. Now, $H^1(S,\mathcal{O}_S)=0$, so that D restricts to a complete linear system of degree 7 on A. The linear system on A given by $|\Sigma_{i=1}^9(E_i\cap A)-(H\cap A)|$ is of degree 5 and thus has a section consisting of five points $\{q_j\}_{j=1}^5$ on A; since A is general in the linear system we may assume that none of these three q_j's are collinear.

Let S' be the surface obtained by blowing up $\mathbf{P}^2$ at these five points and F_j the corresponding exceptional divisors. We have an embedding of S' in $\mathbf{P}^4$ by the linear system $|3H'-\Sigma_{j=1}^5 F_j|$, where H' is the pullback to S' of a general line in $\mathbf{P}^2$. It is well known that this surface is the complete intersection of two quadrics in $\mathbf{P}^4$ (see [1]).

Let A' be the strict transform to S' of the curve A in S; then there is a natural isomorphism between A and A'. Furthermore, by the choice of q_i's, we see that the embedding of A' in $\mathbf{P}^4$ is by the same linear system as the one that embeds A as a hyperplane section of S in $\mathbf{P}^5$; thus we may identify this $\mathbf{P}^4$ with the hyperplane in $\mathbf{P}^5$ which cuts out A in S. The line bundle $\mathcal{O}_{S'}(-A')\otimes\mathcal{O}_{\mathbf{P}^4}(n)$ is generated by global sections for $n\geqslant 3$ (see Nagata *loc. cit.*). For $n=2$ this is $\mathcal{O}_{S'}(2H'-\Sigma_{j=1}^5 F_j)$ which has exactly one section Q, a line in $\mathbf{P}^4$. The union $A'\cup Q$ is then defined by quadrics in $\mathbf{P}^4$ and A' is defined by cubics. Thus we have a surjection

$$\mathcal{O}_{\mathbf{P}^4}(-2)^{\oplus 3}\oplus\mathcal{O}_{\mathbf{P}^4}(-3)\to I_{A'/\mathbf{P}^4}=I_{A/\mathbf{P}^4}.$$

Since A is a general hyperplane section of S, we have the result. □

With notation as above, let P be the plane spanned by the elliptic curve C. For Q as in the proof above we have $Q=P\cap\mathbf{P}^4$. From this one can see that the net N of quadrics containing S also contains P, and in fact $S\cup P$ is the complete intersection of these three quadrics. Hence we have a sequence

$$0\to\mathcal{O}_S^{\oplus 3}\otimes\mathcal{O}_{\mathbf{P}^5}(-2)\to N^*_{S/\mathbf{P}^5}\to T\to 0$$

where $T=N^*_{C/P}=\mathcal{O}_C\otimes\mathcal{O}_{\mathbf{P}^5}(-3)$. The dual sequence is

$$0\to N_{S/\mathbf{P}^5}\to\mathcal{O}_S^{\oplus 3}\otimes\mathcal{O}_{\mathbf{P}^5}(2)\to\mathcal{O}_C\otimes\mathcal{O}_S(C)\otimes\mathcal{O}_P(C)\to 0.$$

The last surjection induces a map $C\to N$. A point outside the image of this gives a quadric containing S which is smooth along S. Similarly, we take the sequence

$$0\to\mathcal{O}_P(-2)^{\oplus 3}\to N^*_{P/\mathbf{P}^5}\to T'\to 0$$

where $T'=N^*_{C/S}$. The dual sequence is

$$0\to N_{P/\mathbf{P}^5}\to\mathcal{O}_P(2)^{\oplus 3}\to\mathcal{O}_C\otimes\mathcal{O}_S(C)\otimes\mathcal{O}_P(C)\to 0$$

which again induces the same map $C\to N$ and so a point outside the image gives us

a quadric which is smooth along $S \cup P$. Since this is the base locus of N, there is a smooth quadric Y containing S.

Choose such a smooth quadric. Then S is defined by cubic equations in Y, i.e. if $V_1 = \Gamma(Y, I_{S/Y} \otimes \mathcal{O}_{\mathbf{P}^5}(3))$ then we have $V_1 \otimes \mathcal{O}_Y \twoheadrightarrow I_{S/Y} \otimes \mathcal{O}_{P^5}(3)$ is surjective. Further, by adjunction we have $K_Y \cong \mathcal{O}_Y \otimes \mathcal{O}_{\mathbf{P}^5}(-4)$. Now we apply the following lemma

Lemma 9. *Let $S \subset Y$ be a pair of projective varieties, and L a very ample line bundle on Y. If $V_1 = \Gamma(Y, I_{S/Y} \otimes L^{\otimes n})$ generates at stalks then $V = \Gamma(Y, I_{S/Y} \otimes L^{\otimes (n+1)})$ is very ample on $\tilde{Y}$, the blow up of Y along S.*

Proof. The surjection $V_1 \otimes \mathcal{O}_Y \to I_{S/Y} \otimes L^{\otimes n}$ induced by the evaluation map gives an inclusion $\tilde{Y} \subset Y \times \mathbf{P}(V_1)$. The line bundle $M = p_1^* L \otimes p_2^* \mathcal{O}_{\mathbf{P}(V_1)}(1)$ is very ample on $Y \times \mathbf{P}(V_1)$. Restricting this to $\tilde{Y}$ and taking direct image to Y we see that $\Gamma(\tilde{Y}, M|_{\tilde{Y}}) = \Gamma(Y, I_{S/Y} \otimes L^{\otimes (n+1)})$. Hence the result. □

Similarly, we can choose Y to be a smooth cubic containing S and then S is defined by quadric equations in Y, i.e. if $V_1 = \Gamma(Y, I_{S/Y} \otimes \mathcal{O}_{\mathbf{P}^5}(2))$ then we have a surjection $V_1 \otimes \mathcal{O}_Y \to I_{S/Y} \otimes \mathcal{O}_{|P^5}(2)$. Also note that by adjunction we have $K_Y \cong \mathcal{O}_Y \otimes \mathcal{O}_{\mathbf{P}^5}(-3)$ so that we can again apply the above lemma.

Finally, to produce the exceptional curves of the first kind on S we use

Lemma 10. *Let S be the surface obtained by blowing up $\mathbf{P}^2$ at at-least 9 general points. Then S contains infinitely many exceptional curves of the first kind.* □

The proof can be found in [8].

We are now in a position to state and prove

Theorem 11. *Let Y be smooth quadric or a smooth cubic in $\mathbf{P}^4$. The anticanonical bundle K_Y^{-1} of Y is very ample. Let X be the geometric generic divisor in the corresponding linear system. Then, the Griffiths group of X contains a subgroup of infinite rank.*

Proof. A simple dimension count shows that every smooth cubic contains a surface S as in Lemma 8. Since all smooth quadrics are isomorphic, the same is true for quadrics as well. Further, as a consequence of Lemma 8, if $V = \Gamma(Y, I_{S/Y} \otimes K_Y^{-1})$, then we have a surjection

$$V \otimes \mathcal{O}_Y \twoheadrightarrow I_{S/Y} \otimes K_Y^{-1}.$$

Furthermore, by Lemma 9, if $\tilde{Y}$ is the blow up of Y along S, then we have a natural embedding $\tilde{Y} \hookrightarrow \mathbf{P}(V)$. We note that this implies that the map $P = \mathbf{P}(N_{S/Y}^*) \to \mathbf{P}(V)$ is also an embedding. Now Y is simply connected and has $\operatorname{Pic}(Y) = \mathbf{Z}$; as a consequence $\operatorname{Pic}(\tilde{Y}) = \mathbf{Z} \oplus \mathbf{Z}[P]$. Let us adopt the notation $\mathcal{O}_Y(1) = K_Y^{-1}$.

First of all we use Lemma 5 to find X_0 which contains S and is smooth along all the exceptional curves in S. Since there are countable many such curves X_0 is defined over some countably generated field.

Let G be the vector bundle defined by the exact sequence

$$0 \to G^* \to V \otimes \mathcal{O}_S \to N_{S/Y}^* \otimes \mathcal{O}_Y(1) \to 0.$$

We have a map $\mathbf{P}_S(G) \to \mathbf{P}(V^*)$. Since $\mathrm{Pic}(Y) = \mathbf{Z}$, the same is true for any smooth divisor D in the linear system $|K_Y^{-1}|$. Then by the adjunction formula, a smooth divisor in D is of general type; in particular, no smooth D can contain S. From this it follows that the map $\mathbf{P}_S(G) \to \mathbf{P}(V^*)$ is generically finite, let B_1 denote its ramification locus. Let B_2 denote the locus in $\mathbf{P}(V^*)$ consisting of v such that the corresponding hyperplane section X_v has singularities outside S, and $B = B_1 \cup B_2$.

By Lemma 10 and S has infinitely many exceptional curves of the first kind. We need to choose among these, curves $E \subset S$ such that the image of $\mathbf{P}_E(G|_E)$ is not contained in B. There are clearly infinitely many of these. Further, we choose an infinite subcollection $\{E_d\}$ such that, in addition, the images of $\mathbf{P}_{E_d}(G|_{E_d}) \xrightarrow{f_d} \mathbf{P}(V^*)$ are distinct. Since the map $\mathbf{P}_S(N^*_{S/Y}(1)) \to \mathbf{P}(V)$ is an embedding we can apply Lemma 6 to show that f_d's are birational to their images.

Now for each d, we choose a point v_d in the image of f_d, which is not in the image of f_l for any $l \neq d$. Further we may assume that v_d is not in B. Let X_d be the corresponding hyperplane section of Y; then $(X_d)_{\mathrm{sing}}$ is a finite collection of ordinary double points lying in S. We apply Lemmas 5 and 7 to conclude that X_d satisfies conditions (1)–(3) listed at the end of § 2.

Let X'_d denote the strict transform of X_d in $\tilde{Y}$; this is the small resolution of X_d. Since it is a hyperplane section of $\tilde{Y}$ it has $\mathrm{Pic}(X'_d) = \mathbf{Z} \oplus \mathbf{Z}[S'_d]$; where $S'_d = P \cap X'_d$ is the strict transform of S in X'_d. The result of blowing up the finitely many exceptional curves of the map $S'_d \to S$ in X'_d is $\tilde{X}_d$, which is the blow up of the finitely many ordinary double points of X_d. From this we see that condition (4) of § 2 is also satisfied.

Note that $\mathbf{Hilb}(X; Y)$ is just the projective space $|K_Y^{-1}|$. Let A_0 be a curve in $\mathbf{Hilb}(X; Y)$ joining X to X_0. We use the second part of Lemma 5 to construct infinitely many rigid rational curves in X, by deforming along A_0 all the exceptional curves of the first kind in S. For each d we choose a curve B_d in $\mathbf{P}(V^*)$ joining X_d to X_0 which is not entirely contained in any of the divisors $\mathbf{P}_{E_d}(G|_{E_d})$ or in B. We may choose a deformation A_d of $A_0 \cup B_d$ in $\mathbf{Hilb}(X; Y)$. The above construction ensures that deformations along the different A_d's give the same collection of rigid rational curves in X. Now we can apply the argument following Lemma 2 to conclude that the classes e_d (with notation as in § 2) generate a subgroup of infinite rank in the intermediate Jacobian $J(X)$ of X.

As a final point, note that $H^3(Y, \mathbf{Z}) = 0$ and thus by well known arguments (as in [6]), the abelian part of $J(X)$ is zero. This implies that this infinite rank subgroup is actually contained in the Griffiths group. □

Remark. Constructions similar to the one above will also allow us to conclude the theorem in the case where Y is $\mathbf{P}^4$, thereby giving the original results of C H Clemens.

Appendix A. Bertini type results

Let X be a smooth variety and Y be a smooth subvariety of codimension r and L be a line bundle such that $\mathscr{I}_{Y/X} \otimes L$ is generated by global sections. We wish to understand the singularities of the general global section of $\mathscr{I}_{Y/X} \otimes L$.

Let $\mathscr{F}$ be the coherent sheaf on X defined by the exact sequence

$$0 \to L^{-1} \to \Gamma(X, \mathscr{I}_{Y/X} \otimes L)^* \otimes \mathcal{O}_X \to \mathscr{F} \to 0.$$

Then $\mathbf{P}_X(\mathscr{F}) \subset X \times \mathbf{P}(\Gamma(X, \mathscr{I}_{Y/X} \otimes L)^*)$ is the incidence locus between sections of $\mathscr{I}_{Y/X}$ and their zero sets.

For any $y \in Y$ we can pick sections $f_1, \dots, f_r \in \Gamma(X, \mathscr{I}_{Y/X} \otimes L)$ which define Y in a neighbourhood of y. The remaining sections can then be expressed as linear combinations of the f_i's,

$$g_i = \sum_{j=1}^{r} a_{i,j} f_j \text{ where } i = 1, \dots, l.$$

Thus, the first homomorphism of the above sequence can be rewritten in a neighbourhood of y as

$$\mathcal{O}_{X,y} \to (\oplus_{j=1}^{r} \mathcal{O}_{X,y} F_j) \oplus (\oplus_{i=1}^{l} \mathcal{O}_{X,y} G_i)$$
$$1 \mapsto \sum_{j=1}^{r} f_j F_j + \sum_{i=1}^{l} g_i G_i$$

where $F_1, \dots, F_r, G_1, \dots, G_l$ is the basis of $\Gamma(X, \mathscr{I}_{Y/X} \otimes L)^*$ which is dual to $f_1, \dots, f_r$, $g_1, \dots, g_l$. Put $H_j = F_j + \Sigma_{i=1}^{l} a_{i,j} G_i$ so that the above homomorphism can be written as

$$\mathcal{O}_{X,y} \to (\oplus_{j=1}^{r} \mathcal{O}_{X,y} H_j) \oplus (\oplus_{i=1}^{l} \mathcal{O}_{X,y} G_i)$$
$$1 \mapsto \sum_{j=1}^{r} f_j H_j$$

where $f_1, \dots, f_r$ define Y, a smooth subvariety of codimension r in a neighbourhood of y and may thus be thought of as "coordinates".

Thus, we have

$$\mathbf{P}_X(\mathscr{F}) = \mathrm{Proj}\left(\frac{\mathcal{O}_{X,y}[H_1, \dots, H_r, G_1, \dots, G_l]}{(\Sigma_{j=1}^{r} f_j H_j)}\right)$$

which can thus be expressed as the union of the affine open pieces of two types

1. The regular pieces are, for each j between 1 and r

$$\mathrm{Spec}\left(\frac{\mathcal{O}_{X,y}[H_1/H_j, \dots, H_r/H_j, G_1/H_j, \dots, G_l/H_j]}{(f_j + \Sigma_{k \neq j} f_k H_k/H_j)}\right)$$

2. The singular pieces are, for each i between 1 and l

$$\mathrm{Spec}\left(\frac{\mathcal{O}_{X,y}[H_1/G_i, \dots, H_r/G_i, G_1/G_i, \dots, G_l/G_i]}{(\Sigma_{j=1}^{r} f_j H_j/G_i)}\right)$$

which is an ordinary double singularity along the locus defined by the vanishing of $f_1, \dots, f_r$ and $H_1/G_i, \dots, H_r/G_i$.

Thus the singular locus of $\mathbf{P}_X(\mathscr{F})$ is smooth of dimension $\dim X - r + l - 1$. The dimension of $\mathbf{P}(\Gamma(X, \mathscr{I}_{Y/X} \otimes L)^*)$ is $r + l - 1$. Therefore, if $\dim X < 2r$ then the general element of $\Gamma(X, \mathscr{I}_{Y/X} \otimes L)$ defines a smooth divisor in X containing Y. If $\dim X = 2r$, then the general element as above has finitely many ordinary double points along Y and is smooth outside.

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Self-dual connections, hyperbolic metrics and harmonic mappings on Riemann surfaces

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Abstract. The Sampson-Wolf model of Teichmüller space (using harmonic mappings) is shown to be exactly the same as the more recent Hitchin model (utilizing self-dual connections). Indeed, it is noted how the self-duality equations become the harmonicity equations. An interpretation of the modular group action in this model is mentioned.*

Keywords. Teichmüller space; self-dual connection; hyperbolic metric; harmonic maps.

1. Introduction

Let M be a compact Riemann surface of genus $g (g \geqslant 2)$ with hyperbolic (Poincaré) metric on M denoted by σ. We will consider (M, σ) as the base point of the Teichmüller space $T(M) = T_g$. As usual, $T(M)$ parametrizes all the possible hyperbolic metrics (constant negative curvature -4) on the smooth surface M up to pullbacks by diffeomorphisms homotopic to the identity. The canonical bundle on (M, σ) will be called K.

In his Stanford thesis, Wolf [5], following on results of Sampson [4], discovered a natural homeomorphism of the Teichmüller space onto the full vector space, $H^0(M, K^2)$, of holomorphic quadratic differentials on the base Riemann surface (M, σ). The method is to look at the unique harmonic diffeomorphism $w = w(\sigma, \rho)$, homotopic to the identity, from (M, σ) to (M, ρ) where ρ is another hyperbolic metric on the smooth surface M. The $(2, 0)$ part of the pullback of ρ by w is a *holomorphic* quadratic differential on the base surface by virtue of w being a *harmonic* diffeomorphism. Wolf associates this quadratic differential to the point of $T(M)$ represented by (M, ρ). Thus one has the homeomorphism

$$\mathscr{W}: T(M) \rightarrow H^0(M, K^2). \tag{1}$$

Of course, $\mathscr{W}$ is not a complex analytic parametrization of T_g since it is well-known

* Acknowledgement: I noted the above identity between Wolf's model and Hitchin's model in 1990. Some experts in the area have pointed out to me that this relationship was privately noted by other mathematicians before me; in particular M Wolf knew it, and communicated it to N J Hitchin, and it was also known to J Eells—and surely others. However, this cute fact is not in published form anywhere to my knowledge, so I think it may be worthwhile to point it out to the larger community of geometers and analysts in ths field.

that holomorphically T_g is a bounded domain. See Nag [3] for the basic facts needed from Teichmüller theory.

Also recently, using self-dual connections on the compact Riemann surface M, Hitchin [1, 2] has given a method to describe the various possible hyperbolic metrics on the smooth surface M. The choice of the self-dual connection, and hence of the corresponding hyperbolic metric, depends only on the choice of an arbitrary holomorphic quadratic differential on M. Thus Hitchin's result again provides a natural parametrization of the Teichmüller space $T(M)$ by the full vector space of holomorphic quadratic differentials $H^0(M, K^2)$. Namely one gets Hitchin's homeomorphism

$$\mathscr{H}: H^0(M, K^2) \to T(M). \tag{2}$$

Our main remark is that Hitchin's model of Teichmüller space as a homeomorphic copy of the vector space $H^0(M, K^2)$ is precisely that discovered by Sampson and Wolf. Namely *the mappings $\mathscr{W}$ and $\mathscr{H}$ are simply inverses to each other.* In fact, we show by a simple change of variables how the self-duality equation becomes the main equation studied by Wolf.

2. Hitchin's method in summary

In the present context, the equation that Hitchin uses to describe the relevant "self-dual connections", depends on the choice of an arbitrary $q(\in H^0(M, K^2))$, and produces a certain hermitian metric $h = h(q)$ on the Riemann surface M as the solution. The self-duality equation asks for this self-dual hermitian metric, h, on M having curvature 2-form $F(h)$ and Kahler form $\omega(h)$ satisfying:

$$F(h) = -2(1 - \|q\|^2)\omega(h)^2. \tag{3}$$

Here $\|q\|^2 = |q|^2/h^2$, represents the squared-norm of q as a function on M with respect to the sought-for metric h. In local holomorphic coordinates z on the base Riemann surface M this reads as follows:

Let $h = h(z)\mathrm{d}z \otimes \mathrm{d}\bar{z}$ and $q = q(z)\mathrm{d}z^2$. Then $\omega(h) = h(z)\mathrm{d}z \wedge \mathrm{d}\bar{z}$, and $F(h) = -\Delta \log h.$ $\mathrm{d}z \wedge \mathrm{d}\bar{z}$. So (3) becomes:

$$\Delta \log h = 2(1 - |q|^2/h^2)h \tag{4}$$

with

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2, (z = x + iy).$$

Remark. Notice that if q is the zero quadratic differential then the conformal (hermitian) metric h has constant negative curvature -4, (as is obvious from (4)). In this case, h is itself the Poincaré metric of M.

Remark. We refer to Hitchin [1, 2] for the reason why (3) arises as the self-dual connections equation for a certain vector bundle over M.

3. The main idea

The interesting discovery by Hitchin is that for arbitrary choice of the holomorphic quadratic differential q on M one can produce a *new hyperbolic metric* on M, called $h^* = h^*(q)$, via the formula

$$h^* = q + (h + q\bar{q}/h) + \bar{q}. \tag{5}$$

This h^* is, indeed, of constant negative curvature -4, again. Hitchin's homeomorphism maps q to the Teichmüller point represented by $(M, h^*(q))$.

Thus h^* is a hyperbolic Riemannian metric on M which is smooth with respect to the original complex structure of M, but is not hermitian with respect to that complex structure for any non-zero choice of the quadratic differential q. Indeed, note that h^* has been written in (5) above in its standard type decomposition into $(2,0)+(1,1)+(0,2)$ parts on M. Because of the presence of non-vanishing $(2,0)$ and $(0,2)$ parts, (the q and its conjugate respectively), this h^* *cannot* be a hermitian (conformal) metric for M.

Note. For some choices of q, however, h^* will produce the same conformal structure on M as the original one, up to diffeomorphism. These quadratic differentials q will be precisely the ones that are Teichmüller modular group translates of the zero differential in this Sampson–Wolf–Hitchin model of Teichmüller space $T(M)$. It would be quite interesting to investigate the nature of these q's that are modular equivalent to zero.

Proof that $\mathscr{W} = \mathscr{H}^{-1}$. Consider the identity diffeomorphism $1:(M,\sigma)\to(M,h^*(q))$. Equation (5) shows that $\rho = h^*(q)$ is a hyperbolic metric such that the $(2,0)$ part of the pullback $1^*(\rho)$ is the holomorphic quadratic differential q on the base surface. Thus 1 must be the unique harmonic diffeomorphism in its homotopy class, and the definition of Wolf's map $\mathscr{W}$ shows that $\mathscr{W}(\rho) = q$, as needed. ■

Remark. Hitchin has used the idea above in § 11 of his paper [1].

4. The equations of self-duality and harmonicity

In Wolf's work the equation for the harmonic diffeomorphism $w = w(\sigma, \rho)$ is studied in terms of the "holomorphic energy density" function $H = H(w)$ on the base (M, σ):

$$\mathscr{H} = \{\rho(w(z))/\sigma(z)\}\cdot|w_z|^2 \tag{6}$$

H satisfies the p.d.e.:

$$(1/\sigma)\Delta \log H = 2(H - |q|^2/(\sigma^2 H)) - 2 \tag{7}$$

(see [5] Chapter I § 2). It is not surprising that Hitchin's model of T_g coincides with Wolf's since the self-duality eq. (3) or (4) is easily related to eq. (7) of harmonicity. In fact we have:

PROPOSITION

The energy density function H for the harmonic diffeomorphism w is strictly positive on M, and is given by the ratio $H = h/\sigma$, where h is the "self-dual" hermitian metric determined by (3).

Proof. Set $h = \sigma H$ and substitute in Hitchin's equation (3) or (4). A short calculation (which uses the fact that σ is a hyperbolic metric) shows that H satisfies (7), as required. The fact that H is positive is well-known (see [5]). ∎

From the above Proposition we can now write the expression for the new hyperbolic metric $\rho = h^*(q)$ from (5):

$$\rho = h^* = q + \sigma(H + |q|^2/(\sigma^2 H)) + \bar{q} \tag{8}$$

utilizing the solution H of Wolf's equation (7). This expression for ρ can actually be seen in [5] Chapter II—thus proving again the identity of the two methods of describing all the hyperbolic structures on M.

In concluding this note we remark that it is interesting to see the ubiquitous role played by the Laplace operator in the various elliptic p.d.e.'s that describe the hyperbolic uniformization of Riemann surfaces. Thus, we see its presence in (4), again in (7), as well as in the classical Liouville equation for hyperbolic metrics.

See [6] for further related work by Wolf.

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Subordination properties of certain integrals

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Abstract. Let $B_1(\mu,\beta)$ denote the class of functions $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ that are analytic in the unit disc Δ and satisfy the condition $\mathrm{Re}\, f'(z)(f(z)/z)^{\mu-1}>\beta$, $z\in\Delta$, for some $\mu>0$ and $\beta<1$. Denote by $S^*(0)$ for $B_1(0,0)$. For μ and c such that $c>-\mu$, let $F=I_{\mu,c}(f)$ be defined by

$$F(z)=\left[\frac{\mu+c}{z^c}\int_0^z f^{\mu}(t)t^{c-1}\,\mathrm{d}t\right]^{1/\mu},\quad z\in\Delta.$$

The author considers the following two types of problems:

(i) To find conditions on μ, c and $\rho>0$ so that $f\in B_1(\mu,-\rho)$ implies $I_{\mu,c}(f)\in S^*(0)$;
(ii) To determine the range of μ and $\delta>0$ so that $f\in B_1(\mu,-\delta)$ implies $I_{\mu,o}(f)\in S^*(0)$;

We also prove that if f satisfies $\mathrm{Re}(f'(z)+zf''(z))>0$ in Δ then the nth partial sum f_n of f satisfies $f_n(z)/z\prec -1-(2/z)\log(1-z)$ in Δ. Here, $\prec$ denotes the subordination of analytic functions with univalent analytic functions. As applications we also give few examples.

Keywords. Differential subordination; univalent star-like; convex function.

1. Introduction

Let A denote the class of functions f, $f(z)=z+a_2z^2+\cdots$, that are analytic in the unit disc $\Delta=\{z:|z|<1\}$. For a given real number, $\beta<1$, let $S^*(\beta)$ and $K(\beta)$ represent the subclasses of A consisting of star-like functions of order β and convex functions of order β, respectively. Let $B_1(\mu,\beta)$ be such that

$$\mathrm{Re}\, f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}>\beta,\quad z\in\Delta$$

for some $\mu>0$ and $\beta<1$; and that let $R(\beta)\equiv B_1(1,\beta)$. For $0\leqslant\beta<1$ functions in these classes are in fact univalent in Δ [1].

For $f\in A$, and μ and c such that $\mu+c>0$, let $F=I_{\mu,c}(f)$ be defined by

$$F(z)=\left[\frac{\mu+c}{z^c}\int_0^z f^{\mu}(t)t^{c-1}\,\mathrm{d}t\right]^{1/\mu},\quad z\in\Delta.\tag{1}$$

Many authors have studied this operator in various situations [1, 8, 10]. However from a more general result obtained in [4], the author as a special case concerning

Alexander and Libera transforms established that

$$I_{0,1}(R(-\delta_1)) \subset S^*(0), \quad \text{for } \delta_1 = 0{\cdot}35\ldots; \tag{2}$$

$$I_{0,1}(R(-\delta_2)) \subset S^*(\beta), \quad \text{for } \delta_2 = 0{\cdot}29\ldots; \tag{3}$$

$$I_{1,1}(R(-\delta_3)) \subset S^*(0), \quad \text{for } \delta_3 = 0{\cdot}09\ldots; \tag{4}$$

$$I_{1,1}(R(-\delta_1)) \subset S^*(\beta), \quad \text{for } \delta_4 = 0, \tag{5}$$

where β is the positive root of a cubic polynomial, and it may be observed that the bounds appearing in the above inclusions are not the best possible ones.

These relations actually improve the earlier results of Mocanu [3], and Singh and Singh [11] who respectively proved (4) and (2) with $\delta_1 = -1/4$ and $\delta_3 = 0$. Further, the question concerning the sharpness of the above implications is still open. However a better bound can be found. For example the author [6] recently described a method for improving the bound for (3). Although the members of a class R_1, $R_1 = \{f \in A: |f'(z) - 1| < 1\}$, are univalent and bounded, the author [5] demonstrated a function $f_0 \in R_1$ such that $f_0 \notin S^*$. In spite of this observation, in the present note, our methods in fact further yield an interesting generalization from $I_{1,c}(R(\beta)) \subset S^*(0)$ to $I_{\mu,c}(B_1(\mu, \beta)) \subset S^*(0)$ concerning the integral operator defined by (1).

2. Preliminaries and main results

If f and g are analytic in Δ, we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if g is univalent in Δ, $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

We are going to prove the following results.

Theorem 1. *Let*

$$L_{\mu,c}(z) = -1 + 2\int_0^1 \frac{\mathrm{d}t}{1 - zt^{1/(\mu+c)}}, \quad M = [1 - L_{\mu,c}(-1)][1 - L_{\mu,0}(-1)],$$

$$N = [1 - L_{\mu,0}(-1)]^2, \quad S = \frac{2\mu + 1}{c^2}[M^2 + 2M(\mu + c)] - 3N,$$

$$T = -2\left(\frac{2\mu + 1}{c^2}\right)[(1 - M)(M + \mu + c)] - 6N,$$

$$W = \frac{2\mu + 1}{c^2}(1 - M)^2 - 3N \quad \text{and} \quad \rho = \frac{-T - (T^2 - 4SW)^{1/2}}{2S}.$$

If $f \in A$ satisfies

$$\mathrm{Re}\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > -\rho$$

then for $F(z)/z \neq 0$ in Δ,

$$F = I_{\mu,c}(f) \in S^*(0)$$

for $0 < \mu \leqslant 2/\sqrt{3}$ and $-\mu < c \leqslant \sqrt{(2\mu + 1)/3}$.

An adaptation of the same method would give us the following theorem to deal with the case $c=0$.

Theorem 2. *Let $\mu>0$ be determined from*

$$N=[1-L_{\mu,0}(-1)]^2 \geqslant 1.$$

Then for this range of μ, we, for $(I_{\mu,0}(f))(z)/z \neq 0$ in Δ, have

$$\mathrm{Re}\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > -\delta = -\frac{N-1}{N+2\mu} \text{ implies } I_{\mu,0}(f) \in S^*(0).$$

To prove our theorems, we require the following lemmas:

Lemma A. [2] *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that the function $\psi:\mathbb{C}^2\times\Delta\to\mathbb{C}$ satisfies the condition $\psi(ix,y;z)\notin\Omega$, for all real $x, y\leqslant -(1+x^2)/2$ and all $z\in\Delta$. If the function p defined by $p(z)=1+c_1z+c_2z^2+\ldots$ is analytic in Δ and if $\psi(p(z),zp'(z);z)\in\Omega$, then* Re $p(z)>0$ *in Δ.*

Using the arguments of [6, lemma 1] (see also [4]) it is clear that the following more general lemma holds.

Lemma B. If p is analytic in Δ with $p(0)=\exp(i\gamma)$, (γ real and fixed), $\alpha\in\mathbb{C}$ with $\mathrm{Re}\,\alpha \geqslant 0 (\alpha\neq 0)$, *$\beta<1$ and that*

$$Re\{p(z)+\alpha zp'(z)\} > \beta\cos\gamma, \quad z\in\Delta$$

then

$$Re\,p(z) > \beta\cos\gamma + (1-\beta)\cos\gamma[2\delta(Re\,\alpha)-1], \quad z\in\Delta$$

where δ is given by

$$\delta=\delta(Re\,\alpha)=\int_0^1 \frac{dt}{1+t^{Re\,\alpha}}$$

is an increasing function of $Re\,\alpha$ and $(1+Re\,\alpha)/(1+2Re\,\alpha)\leqslant\delta<1$. The estimate is sharp in the sense that the bound cannot be improved.

Example 1. If we let $R_\gamma(\beta)$ to be the class of analytic functions

$$f(z)=\exp(i\gamma)z+a_2z^2+\ldots.., \quad (\gamma \text{ real and fixed})$$

defined on Δ, so that $\mathrm{Re}\,f'(z)>\beta\cos\gamma$, then

$$f\in R_\gamma(\beta) \text{ implies } f(z)/z \prec \beta\cos\gamma + i\sin\gamma$$
$$+(1-\beta)\cos\gamma[-1-(2/z)\log(1-z)], \quad z\in\Delta.$$

Lemma C. [6, Corollary 1] *Let p be analytic in Δ, with $p(0)=1$. Suppose that an analytic function λ on Δ satisfies*

$$|Im\,\lambda(z)| \leqslant \sqrt{3}(Re\,\lambda(z)-\sqrt{3}/2), \quad z\in\Delta \tag{6}$$

then

$$Re\{p(z)+\lambda(z)zp'(z)\}>0 \tag{7}$$

implies $|arg\,p(z)|<\pi/3$, $z\in\Delta$.

It seems fitting to add the following result at this point, since it is also related to our further investigation.

COROLLARY

Let $c \neq 0$ be a complex number and p and h be analytic in Δ, with $p(z)\cdot h(z) \neq 0$,

$$p(z) + \frac{zp'(z)}{c} = h(z) \tag{8}$$

and

$$\left|arg\left(\frac{p(z)}{cp(z)+zp'(z)} - \frac{\sqrt{3}}{2}\right)\right| < \frac{\pi}{3}, \quad z\in\Delta. \tag{9}$$

Let g be analytic in Δ, $g(0)=1$ with $Re\, g(z) > 0$, for $z\in\Delta$. If $G = I(g)$ is defined by

$$G(z) = cz^{-c}p(z)^{-1}\int_0^z g(t)t^{c-1}h(z)\mathrm{d}t, \quad z\in\Delta \tag{10}$$

then G is analytic in Δ, $G(0)=g(0)$ and $|arg\, G(z)| < \pi/3$ for Δ.

Proof. Since $z=0$ in (8) and (9) gives $p(0)=h(0)$ and $|\arg(1/c - \sqrt{3}/2)| < \pi/3$, the conditions on p, h and g imply that G is analytic in Δ, $G(0)=1$. Introduce $\lambda(z) = p(z)/(cp(z)+zp'(z))$. Since $\mathrm{Re}\, g(z) > 0$, by differentiating (10) we obtain

$$\mathrm{Re}(G(z) + \lambda(z)zG'(z)) = \mathrm{Re}\, g(z) > 0, \quad z\in\Delta.$$

Hence conditions (6) and (7) of Lemma C is satisfied with $p = G$, and so we conclude that $|\arg G(z)| < \pi/3$. □

Example 2. If we take $p(z) = \exp(\gamma z)$ (and hence $h(z) = (1+(\gamma/c)z)\exp(\gamma z)$, we obtain: if g is analytic in Δ with $g(0)=1$ then

$$\mathrm{Re}\, g(z) > 0 \text{ implies} \left|\arg\left(cz^{-c}\exp(\gamma z)\int_0^z g(z)t^{c-1}\exp(\gamma t)(1+(\gamma/c)t)\mathrm{d}t\right)\right| < \pi/3,$$

provided

$$\left|\arg\left(\frac{1}{c+\gamma z} - \frac{\sqrt{3}}{2}\right)\right| < \pi/3 \text{ for } z\in\Delta.$$

As an immediate application of this result and Lemma *B*, it follows that the function q_α defined by (see also [6])

$$q_\alpha(z) = \frac{1}{\alpha}z^{-1/\alpha}\int_0^z \frac{1+t}{1-t}t^{1/\alpha - 1}\mathrm{d}t = -1 + 2\int_0^1 \frac{\mathrm{d}t}{1-zt^\alpha}, \quad z\in\Delta \tag{11}$$

satisfies

$$\mathrm{Re}\, q_\alpha(z) > -1 + 2\int_0^1 \frac{\mathrm{d}t}{1+t^{\mathrm{Re}\,\alpha}}, \text{ provided Re } \alpha > 0; \tag{12}$$

and

$$|\arg q_\alpha(z)| < \pi/3, \quad \text{provided } |\arg(\alpha - \sqrt{3}/2| < \pi/3. \tag{13}$$

Lemma D. If $p(z)$ is analytic in Δ, $p(0)=1$, and $\operatorname{Re} p(z) > 1/2$, $z \in \Delta$, then for any function g, analytic in Δ, the function $p * g$ takes values in the convex hull of the image of Δ under g.

Using Lemma D, which directly follows by using the Herglotz's representation, we draw the following example:

Example 3. For

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A, \quad \text{let } f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$

Then

$$f_n(z)/z = p(z) * (-1 + 2\varphi(z)),$$

where * is meant for Hadamard product (or convolution),

$$p(z) = 1 + \frac{1}{2} \sum_{k=2}^{\infty} k^2 a_k z^{k-1} \quad \text{and } \varphi(z) = \left(1 + \sum_{k=2}^{n} k^{-2} z^{k-1}\right).$$

If $f \in A$ satisfies

$$\operatorname{Re}(f'(z) + zf''(z)) > 0, \quad z \in \Delta,$$

then the function p defined above shows that $\operatorname{Re} p(z) > 1/2$ in Δ. From the fact that [7]

$$\operatorname{Re}\left(g(z) = 1 + \sum_{k=2}^{n} k^{-1} z^{k-1}\right) > 1/2, \quad z \in \Delta$$

we from (11) easily deduce that

$$\varphi(z) = \frac{1}{z} \int_0^z g(t)\,\mathrm{d}t \prec \tfrac{1}{2} + \tfrac{1}{2}\left[-1 + 2\int_0^1 \frac{\mathrm{d}t}{1 - zt}\right],$$

or, equivalently

$$2\varphi(z) - 1 \prec E(z), \quad z \in \Delta,$$

where $E(z) = -1 - (2/z)\log(1-z)$, and $E(\Delta) \subset \{\omega : \operatorname{Re}\omega > 2\ln 2 - 1\} \cap \{w : |\arg \omega| < \pi/3\} \cap \{\omega : |\operatorname{Im}\omega| < \pi\}$.

This from Lemma D proves that if $zf' \in R(0)$ then for every $n \geqslant 1$ we have

$$f_n(z)/z \prec E(z), \quad z \in \Delta.$$

Remark. In 1928, [12] Szegö proved that if $f \in K(0)$ ($S^*(0)$), then all sections f_n are convex (starlike with respect to origin) in $|z| < 1/4$. Recently in [9], Ruscheweyh obtained these as very special cases of more general results.

Proof of the Theorem 1. Consider the function P defined by

$$P(z) = F'(z)(F(z)/z)^{\mu - 1}, \quad z \in \Delta.$$

Then P is analytic in Δ, $P(0) = 1$ and therefore using (1) and a little calculation, we

have

$$P(z)+\frac{zP'(z)}{\mu+c}=f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}, \quad z\in\Delta. \tag{14}$$

Since $f\in B_1(\mu,-\rho)$, (14) gives that,

$$P(z)\prec -\rho+(1+\rho)\,L_{\mu,c}(z). \tag{15}$$

Thus (15) becomes, if we use either (12) or Lemma B,

$$\operatorname{Re} F'(z)\left(\frac{F(z)}{z}\right)^{\mu-1} > -\rho+(1+\rho)\,L_{\mu,c}(-1), \quad z\in\Delta. \tag{16}$$

If we substitute $\beta=-\rho+(1+\rho)\,L_{\mu,c}(-1)$ and note that we can now rewrite (16) as $F\in B_1(\mu,\beta)$, we at once see from Lemma B that

$$\left(\frac{F(z)}{z}\right)^{\mu}\prec\beta+(1-\beta)\,L_{\mu,0}(z),$$

holds in Δ. From (11), (12) and (13) it can be observed that the function Q defined by $Q(z)=(F(z)/z)^{\mu}$ satisfies

$$Q(\Delta)\subset\Omega_1\cap\Omega_2, \quad \text{provided } 0<\mu\leqslant 2/\sqrt{3}, \tag{17}$$

where

$$\Omega_1=\{\omega\in\mathbb{C}:\operatorname{Re}\omega>\beta+(1-\beta)\,L_{\mu,0}(-1)\} \quad \text{and}$$

$$\Omega_2=\{\omega\in\mathbb{C}:|\arg(\omega-\beta)|<\pi/3\}.$$

We now apply Lemma A. If we denote

$$p(z)=zF'(z)/F(z)$$

then p is analytic in Δ, $p(0)=1$ and from (1) it follows easily that

$$\frac{Q(z)}{\mu+c}[zp'(z)+\mu p^2(z)+cp(z)]=f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}.$$

Thus if $f\in B_1(\mu,-\rho)$, we get from the above identity that

$$\{\psi(p(z),zp'(z);z):|z|<1\}\subset\Omega=\{\omega\in\mathbb{C}:\operatorname{Re}\omega>-\rho\},$$

where

$$\psi(r,s;z)=Q(z)[s+\mu r^2+cr]/(\mu+c).$$

To prove our theorem, it suffices to show, by Lemma A that

$$\psi(ix,y;z)\notin\Omega \tag{18}$$

for all real x, $y\leqslant-(1+x^2)/2$ and all $z\in\Delta$.

If we set $\operatorname{Re}Q(z)=U$ and $\operatorname{Im}Q(z)=V$, we see that

$$\begin{aligned}\operatorname{Re}\psi(ix,y;z)&=[U(y-\mu x^2)-Vcx]/(\mu+c)\\ &\leqslant-[U(1+2\mu)x^2+2Vcx+U]/(2(\mu+c)),\end{aligned}$$

when x is real and $y \leqslant -(1+x^2)/2$. So (18) holds provided $Q = U + iV$ satisfies

$$\left[\frac{U-\rho(\mu+c)}{\rho(\mu+c)}\right]^2 - \left[\frac{V}{(2\mu+1)^{1/2}(\mu+c)\rho/|c|}\right]^2 \geqslant 1. \tag{19}$$

It is easy to check that

$$\beta + (1-\beta)\, L_{\mu,0}(-1) = 1 - (1+\rho)M \quad (= U_0, \text{say}),$$

and

$$\sqrt{3}(1-\beta) = \sqrt{3}(1+\rho)(1 - L_{\mu,0}(-1)) \quad (= V_0, \text{say}).$$

So finally, (19) holds if

$$\left[\frac{U_0-\rho(\mu+c)}{\rho(\mu+c)}\right]^2 - \left[\frac{V_0}{(2\mu+1)^{1/2}(\mu+c)\rho/|c|}\right]^2 = 1,$$

which holds because by the hypothesis ρ is the smallest positive root of the equation

$$S\rho^2 + T\rho + W = 0,$$

where S, T and W are as in the statement of Theorem 1. The proof of Theorem 1 is, therefore, complete. □

Proof of Theorem 2. Using the same technique as in the proof of Theorem 1, to prove Theorem 2, by Lemma A, it is sufficient to show that

$$\psi(ix, y; z) \notin \Omega = \{\omega \in \mathbb{C} : \operatorname{Re}\omega > -\delta\},$$

when x real, $y \leqslant -(1+x^2)/2$ and all $z \in U$; where

$$\psi(r, s; z) = Q(z)(s + \mu r^2)/\mu \quad \text{with } Q(z) = (F(z)/z)^\mu \text{ and } I_{\mu,0}(f) = F$$

and δ is defined in Theorem 2.

By hypothesis $f \in B_1(\mu, -\delta)$ and so this gives $F \in B_1(\mu, -\delta + (1+\delta)\, L_{\mu,0}(-1))$. In view of this and Lemma B, we, from a simple manipulation, easily obtain that $\operatorname{Re} Q(z) > 2\mu\delta$ in Δ.

Thus, we have for real $x, y \leqslant -(1+x^2)/2$ and all $z \in U$,

$$\operatorname{Re}\psi(ix, y; z) \leqslant -(2\mu\delta)/(2\mu) = -\delta,$$

and so the conclusion of Theorem 2 follows. □

Remark. We observe that $\operatorname{Re} f'(z)(f(z)/z)^{\mu-1} > -\delta, (\delta > 0), z \in \Delta$, need not imply the univalence of f in Δ. Further it is also clear from the proofs that the bounds of Theorems 1 and 2 are not the best possible ones. Morever, the method of proof allows us to find better estimation if one is able to determine the best values of δ, $(\delta = \delta(\alpha) > 0)$, and η for which

$$\operatorname{Re}(p(z) + \alpha z p'(z)) > -\delta \text{ implies } |\arg p(z)| < \pi\eta/2, \quad z \in \Delta$$

whenever p is analytic in Δ with $p(0) = 1$.

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The seminormality property of circular complexes

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Abstract. In this paper we prove that the ring $R[X, Y]/(X \cdot Y, Y \cdot X)$ is seminormal, where R is a Cohen–Macaulay normal domain and X, Y are matrices of indeterminates.

Keywords. Seminormal; Pic(R), $U(R)$.

1. Introduction

We prove the seminormality property of the Buchsbaum–Eisenbud variety of circular complexes having coordinate ring $A = (R[X, Y]/(X \cdot Y, Y \cdot X))$, where R is a Cohen–Macaulay normal domain.

These varieties have attracted the attention of many people like Buchsbaum, Eisenbud, Kempf, Huneke, etc. (Prof. M S Narasimhan asked during Indo-USSR conference whether the ring A is seminormal.) Elisabetta Strickland [2] has given quite a clear picture of the components of A and she has proved that each component of this variety is normal. We express the ideals of their intersections in suitable forms and observe that these intersections are reduced. We use homological criteria of seminormality and prove that their unions (of components in suitable order) are seminormal, which ultimately proves that the variety is seminormal.

In § 2, we define seminormality and quote results proved in [2] and [3]. In § 3 we prove the main theorem as stated in the first paragraph.

Before proceeding further we set up few notations.

$X := (X_{ij})_{n_0 \times n_1}$ where X_{ij} is an indeterminate.
$Y := (Y_{ij})_{n_1 \times n_0}$ where Y_{ij} is an indeterminate.
$R[X, Y]$ denotes the R-algebra generated by the entries of X and Y.
$X \cdot Y$ and $Y \cdot X$ denote the sets of entries of the matrices $X \cdot Y$ and $Y \cdot X$ respectively.
$I_r(X)(\text{resp.}\, I_r(Y)) = \{(r+1) \times (r+1) \text{ minors of } X (\text{resp. } Y)\}$
$I(k_1, k_2) = \langle X \cdot Y, Y \cdot X, I_{k_1}(X), I_{k_2}(Y) \rangle$
$(x^0, y^0) \in R^N$, where $N = 2(n_0 \times n_1)$, $x^0 \in M_{n_0 \times n_1}(R)$ and $y^0 \in M_{n_1 \times n_0}(R)$.
$W(k_1, k_2) = \{(x^0, y^0) | x^0 \cdot y^0 = 0,\ y^0 \cdot x^0 = 0,\ rk x^0 \leqslant k_1, rk y^0 \leqslant k_2\}$.
For a given ring $R, R^{[n]} := R[Z_1, \ldots, Z_n]$ where Z_i's are indeterminates.
$U(R)$ = the group of units of R.
Pic(R) = the group of isomorphism classes of invertible modules of R under the binary operation '$\otimes$' (tensor product).

2. Preliminaries

We give an algebraic definition of 'seminormality' which is equivalent to the other notions of seminormality (see [3]).

DEFINITION

A commutative ring R is said to be *seminormal* provided, i) it is reduced. ii) for $a \in Q(R)$, the total quotient ring of R, such that $a^3, a^2 \in R$ then $a \in R$.

Remark. From this definition it is obvious that a normal ring is always seminormal.

For the convenience of the reader we state the necessary results of Strickland [2] and Swan [3].

Theorem 1. (i) *The ring $R[X, Y]/I(k_1, k_2)$ is reduced and has required property that its R-valued points correspond to the set $W(k_1, k_2)$.* (ii) *Furthermore, if the ring R is Cohen–Macaulay normal domain with $k_1 + k_2 \leqslant min(n_0, n_1)$ then the ring $R[X, Y]/I(k_1, k_2)$ is Cohen–Macaulay normal domain and $dim R[X, Y]/I(k_1, k_2) = [(n_0 + n_1) - (k_1 + k_2)] \cdot (k_1 + k_2)$.*

Proof. For a proof refer to [2].

Theorem 2. *Let R be any commutative ring. Then the following properties are equivalent.*

i) *$Pic(R) = Pic R[X_1, \ldots, X_n]$ for some $n \geqslant 1$.*
ii) *$Pic(R) = Pic R[X_1, \ldots, X_n]$ for all n.*
iii) *R_{red} is seminormal.*

Proof. For a proof refer to [3].

3. The main theorem

In this section we state and prove the main theorem. We begin with some lemmas which are required for the proof of the main theorem.

Lemma 1. *Let R be a commutative ring, let J and K be two ideals of R such that R/J and R/K are seminormal. Then $R/J \cap K$ is seminormal if and only if $R/J + K$ is reduced.*

Proof. Consider the following commutative diagram with the canonical maps

$$\begin{tikzcd} R/J \cap K \arrow[r] \arrow[d] & R/J \arrow[d, "\eta_1"] \\ R/K \arrow[r, "\eta_2"] & R/(J+K) \end{tikzcd}$$

This is a cartesian square with surjective maps η_1 and η_2. Therefore one has a Mayer–Vietoris sequences with the following commutative diagram (see [1]).

$$\begin{tikzcd}
U(R/(J\cap K)) \arrow[r] \arrow[d, "f_1"] & U(R/J)\oplus U(R/K) \arrow[r] \arrow[d, "f_2"] & U(R/(J+K)) \arrow[r, "h_1"] \arrow[d, "g_1"] & {} \\
U(R/(J\cap K))^{[n]} \arrow[r] & U(R/J)^{[n]}\oplus U(R/K)^{[n]} \arrow[r] & U(R/J+K)^{[n]} \arrow[r, "h_2"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "h_1"] & \mathrm{Pic}(R/(J\cap K) \arrow[r] \arrow[d, "g_2"] & \mathrm{Pic}(R/J)\oplus \mathrm{Pic}(R/K) \arrow[r] \arrow[d, "g_3"] & {} \\
{} \arrow[r, "h_2"] & \mathrm{Pic}(R/J\cap K)^{[n]} \arrow[r] & \mathrm{Pic}(R/J)^{[n]}\oplus \mathrm{Pic}(R/K)^{[n]} \arrow[r] & {}
\end{tikzcd}$$

where all vertical arrows are injective. Now R/J and R/K reduced implies f_1 is an isomorphism and as the rings are seminormal, we have the isomorphic map g_2 (by theorem 2). Therefore f_2 is an isomorphism if and only if g_1 is. In other words $R/J+K$ is reduced if and only if $R/J\cap K$ is seminormal.

Lemma 2. Let the ring R be a Cohen–Macaulay, normal domain and let I_i denote the ideal $\langle X\cdot Y, Y\cdot X, I_i(X), I_{n_1-i}(Y)\rangle$. Then the ideal $I_0\cap\cdots\cap I_r=\langle X\cdot Y, Y\cdot X, I_r(X)\rangle$ for every $0\leqslant r\leqslant n_1$, where $n_1\leqslant n_0$.

Proof. First we prove the following equality

$$W(r,n_1)= W(0,n_1)\cup W(1,n_1-1)\cup\cdots\cup W(r,n_1-r).$$

We denote the right hand side of the equation by W_r. Take $(x^0,y^0)\in W(r,n_1)$. Then we have $x^0\cdot y^0=0$, $y^0\cdot x^0=0$ and $rkx^0=i\leqslant r$. But $x^0\cdot y^0=0$ implies $rkx^0+rky^0\leqslant n_1$ which implies $rky^0\leqslant n_1-i$ and therefore $(x^0,y^0)\in W(i,n_1-i)$. Conversely, it is obvious that the set $W_r\subseteq W(r,n_1)$. Hence we have $W(r,n_1)= W_r$. Therefore

$$I(W(r,n_1))=I(W_r)=I(W(0,n_1))\cap\cdots\cap(I(W(r,n_1-r)).$$

By theorem (1) this is equivalent to

$$\langle X\cdot Y, Y\cdot X, I_r(X)\rangle=I_0\cap\cdots\cap I_r$$

for $1\leqslant r\leqslant n_1$.

Now we prove the main theorem.

Theorem. *Let R be a Cohen–Macaulay normal domain.*

i) *The ring $R[X, Y]/\langle X\cdot Y, Y\cdot X\rangle$ is seminormal.*
ii) *It has n_1+1 components, each of dimension $n_0\cdot n_1$, where we assume that $n_1\leqslant n_0$.*

Proof. Let $A=R[X, Y]$ and $I=\langle X\cdot Y, Y\cdot X\rangle$ and $I_i=I(i,n_1-i)$. By theorem 1, the ring A/I is reduced and A/I_i is normal hence seminormal. By lemma 2, one has $I=I_0\cap\cdots\cap I_{n_1}$. To prove the theorem, it is enough to prove that, the ring $A/I_0\cap\cdots\cap I_{r-1}$ seminormal implies $A/I_0\cap\cdots\cap I_r$ is seminormal for any $r\leqslant n_1$. Let us denote by J the ideal $I_0\cap\cdots\cap I_{r-1}$ and by K the ideal I_r then $J+K=\langle X\cdot Y, Y\cdot X, I_{r-1}(X), I_{n_1-r}(Y)\rangle$.

Therefore by theorem 1, $A/J+K$ is reduced. Hence by lemma 1 $A/J \cap K$ is seminormal. Thus $R[X,Y]/\langle X\cdot Y, Y\cdot X\rangle$ is seminormal.

Now we have $\langle X\cdot Y, Y\cdot X\rangle = I_0 \cap \cdots \cap I_{n_1}$. Consider the ideals I_i, I_j where $i<j$. Consider the canonical surjective maps $\eta_1: R[X,Y] \to R[X]$ sending all the entries of Y to zero and $\eta_2: R[X,Y] \to R[Y]$ sending all the entries of X to zero. Since $\eta_1(I_i) = \langle I_i(X)\rangle$ and $\eta_1(I_j) = \langle I_j(X)\rangle$, we have $I_i \nsubseteq I_j$. Similarly, as $\eta_2(I_i) = \langle I_{n_1-i}(Y)\rangle$ and $\eta_2(I_j) = \langle I_{n_1-j}(Y)\rangle$ we have $I_j \nsubseteq I_i$. Therefore $I_0, \cdots, I_{n_1}$ are minimal primes of $\langle X\cdot Y, Y\cdot X\rangle$. Moreover, by theorem 1 $\dim R[X,Y]/I_i = n_0 \cdot n_1$.

Hence we conclude that the ring $R[X,Y]/\langle X\cdot Y, Y\cdot X\rangle$ is seminormal with n_1+1 equidimensional, normal, Cohen–Macaulay components.

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papers must normally present results of original work. Critical reviews of portant fields will also be considered for publication. Submission of the typescript be held to imply that it has not been previously published, it is not under sideration for publication elsewhere and that, if accepted, it will not be published where. A paper in applied areas will be considered only on the basis of its thematical content.

escript:

ers should be typed double spaced with ample margin on all sides on white bond er of size 280 × 215 mm. This also applies to the abstract, tables, figure captions the list of references which are to be typed on separate sheets.

e page:

The title of the paper must be short and contain words useful for indexing.
The authors' names should be followed by the names and addresses of the itutions of affiliation.
An *abbreviated running title* of not more than 50 letters and spaces should be given.

tract:

h paper must be accompanied by an abstract describing, in not more than 200 ds, the significant results reported in the paper.

words:

veen 3 and 6 keywords must be provided for indexing and information retrieval.

Text:

paper should preferably start with an introduction in which the results are placed

in perspective and some indication of the methods of proof is given.

1. *Markings*: The copy intended for the printer, should be marked appropriately to make it unambiguous. The following conventions may be followed for the purpose of indicating special characters: (A list of the special characters is included at the end).

a) Indicating characters by underlining

Italics	Single underline
Bold face	Wavy underline
Greek	Blue underline
Fraktur (Gothic)	Red underline
Script	Green underline
Open face	Brown underline

All parameters in equations are normally printed in italics and numerals in upright typeface.

b) In doubtful cases the position of superscripts and subscripts (exponents and indices) should be clearly marked, for example b_{j}, a^{i}, $b^{a^{x}}$, $b_{x_{A}}$, etc.

c) Formulae extending beyond the printed line will be broken by the typesetter at an appropriate place; to avoid any expensive alterations at the proof stage the author may indicate in the manuscript where the formulae should be broken.

d) Various likely ambiguous spots may be explained by pencilled notes in the margin. The following symbols are frequently confused and need to be clarified.

$\circ, o, \upsilon, O, 0$; $\cup, \bigcup, U$; $\times, x, X, \chi, \kappa$; $\vee, \upsilon, v$; $\theta, \Theta, \phi, \varphi, \Phi, \varnothing$; ψ, Ψ; $\varepsilon, \in$;
$a, \alpha, \propto$; B, β; r, γ; $\sigma, 6$; $+, \dagger$; i, ι; a', a^1; the symbol a and the indefinite article a;
also the handwritten Roman letters:
c, C; e, l; I, J; k, K; o, O; p, P; s, S; u, U; v, V; w, W; x, X; z, Z.

2. *Notations and style*: Notation and style should be chosen carefully, keeping the printing aspect in mind. The following table indicates preferred forms for various mathematical usages.

preferred form	instead of	preferred form	instead of
$A^*, \tilde{b}, \gamma', \mathbf{v}$, etc.	$\bar{A}, \hat{b}, \check{\gamma}, \vec{v}$, etc.	$\exp(-(x^2+y^2)/a^2)$	$e^{-((x^2+y^2)/a^2)}$
lim sup, proj lim	$\overline{\lim}$, $\underline{\lim}$		
$f: A \to B$	$A \xrightarrow{f} B$		
$\sum_{n=1}^{\infty}$	$\displaystyle\sum_{n=1}^{\infty}$	$\dfrac{\cos(1/x)}{(a+b/x)^{1/2}}$	$\dfrac{\cos\dfrac{1}{x}}{\sqrt{a+\dfrac{b}{x}}}$

Tables:

All tables must be numbered consecutively in arabic numerals in the order of appearance in the text. The tables should be self-contained and must have a descriptive title.

Figures:

A figure should be included only when it would be substantially helpful to the reader in understanding the subject matter. The figures should be numbered consecutively in arabic numerals in the order of appearance in the text and the location of each figure should be clearly indicated in margin as "Figure 1 here". The figure captions must be typed on a separate sheet.

Line drawings must be in Indian ink on good quality tracing paper. Lines must be drawn sufficiently thick for reduction to a half or third of the original size (0·3 mm for axes and 0·6 mm for curves are suggested).

List of symbols:

For the convenience of the printers, authors should attach to the manuscript the complete list of symbols identified typographically.

References

References should be cited in the text by serial numbers only (e.g. [3]). They should be listed alphabetically by the author's name (the first author's name in the case of joint authorship) at the end of the paper. In describing each reference the following order should be observed: the author's name followed by initials, title of the article, the name of the journal, the volume number, the year and the page numbers. Standard abbreviations of journal titles should be used. It would be worthwhile to cross-check all references cited in the text with the ones given at the end.

A typical reference to an article in a journal would be like:

[1] Narasimhan M S and Ramanan S, Moduli of vector bundles on a compact Riemann surface, *Ann. Math.* **89** (1969) 14–51

A References to a book would be on the following lines:

[8] Royden H, Invariant metrics on Teichmüller space, in: Contributions to analysis (eds) L Ahlfors *et al* (1974) (New York: Academic Press) pp. 393–399

Footnotes:

Footnotes to the text should be avoided if possible. If unavoidable, they should be numbered consecutively, and typed on a separate sheet.

Proofs:

The journal is now typeset by computer photocomposition and only a page-proof is supplied to authors. While the process yields better results, it makes corrections at proof stage very difficult and expensive. Deleting a letter or word in the proof can mean resetting the whole line or paragraph. Similarly, addition of a sentence or paragraph might lead to resetting the whole page and sometimes even the whole article.

Authors are requested to prepare the manuscript carefully before submitting it for publication to minimise corrections and alterations in the proof stage which increase publication costs. The proofs sent to the authors together with the reprint order form must be returned to the editorial office *within two days of their receipt.*

Reprints:

50 reprints of each article will be supplied free of charge.

ATIN LIGHT ROMAN GROUP 1

BCDEFGHIJKLMNOPQRSTUVWXYZ
bcdefghijklmnopqrstuvwxyz 1234567890&
:;!?"-—·*/%()[]£$†‡°ı

ATH SIGNS

⊂ ⊃ × ≃ ∞ ∝ ∈ ∉ □ ○ √ √ ⇌ ⇒ ⇐ ≑ ≠ ···
→↔⊢⊣ ⋖ ⋗ ⊏ ⊕ ⊗ ⊞ ↤ ⊙ ↾ ∧ ∨ ⇝ ¬ ⌐ ⊢
⌟ ⌐ ♭ ⇁ ↿ ⊊ + − ± × = ÷ ∼ → ← ≡ ′″ \
()[]⟨⟩{}⋮∴ ⩽ ⩾ ≦ ≧ ⊆ ⊇ ∩ ∪ ⋂ ⋃ < > ≮ ≯
∀∃ ≅ ≲ ≳ ⊇ ⊆ ≷ ≶ ∓ ⊕ ⊖ ⊗ ≃ ≺ ≻ ⋛ ⋚
↔ ⇔ ≙ ≳ ≲ ◯ ◯ ⊥ ∧ ∨ ¬ ℵ ℶ ⋞ ⋟ ≼ ≽ ↕ ⩽ ⩾ ∮
≪ ≫ ⇆ ⇄ ∅ ♮ ⇋ ⇌ ⊨ ↖ ↘ ↗ ↙ ‰ § ¶

STRONOMICAL SIGNS

♃ ♄ ☊ ♈ ♉ ♍ ♏ ☋ ♂ ♀ Ð Þ þ Ƀ ƀ ʒ ʒ ʓ ʓ

ATIN LIGHT ITALIC GROUP 1

BCDEFGHIJKLMNOPQRSTUVWXYZ
bcdefghijklmnopqrstuvwxyz 1234567890&
:;!?"-—·/%()[]£$†‡°ı*

ATIN BOLD ROMAN GROUP 2

BCDEFGHIJKLMNOPQRSTUVWXYZ
bcdefghijklmnopqrstuvwxyz 1234567890&
:;!?"-—·*/%()[]£$ı

ATIN BOLD ITALIC GROUP 2

BCDEFGHIJKLMNOPQRSTUVWXYZ
bcdefghijklmnopqrstuvwxyz&
:;!?"-——·*/%()[]£$ı

ATIN LIGHT ROMAN GROUP 7

BCDEFGHIJKLMNOPQRSTUVWXYZ
cdefghijklmnopqrstuvwxyz 1234567890
/()[]‡‡‡ min max lim log

ATIN LIGHT ITALIC GROUP 7

BCDEFGHIJKLMNOPQRSTUVWXYZ
cdefghijklmnopqrstuvwxyz

ATH SIGNS GROUP 7

⊃ × ≃ ∞ ∝ ∈ ∉ □ ○ √ √ ⇌ ⇒ ⇐ ≑ ≠ ···
→⊢⊣ ⋖ ⋗ ⊏ ⊕ ⊗ ⊞ ↤ ⊙ ↾ ∧ ∨ ⇝ ¬ ⌐ ⊢
⌟ ♭ ⇁ ↿ ⊊ + − ± × = ÷ ∼ → ← ≡ ′″ \
]⟨⟩{}⋮∴ ⩽ ⩾ ≦ ≧ ⊆ ⊇ ∩ ∪ ⋂ ⋃ < > ≮ ≯
∃ ≅ ≲ ≳ ⊇ ⊆ ≷ ≶ ∓ ⊕ ⊖ ⊗ ≃ ≺ ≻ ⋛ ⋚
⇔ ≙ ≳ ≲ ◯ ◯ ⊥ ∧ ∨ ¬ ℵ ℶ ⋞ ⋟ ≼ ≽ ↕ ⩽ ⩾ ∮
≫ ⇆ ⇄ ∅ ♮ ⇋ ⇌ ⊨

EEK LIGHT UPRIGHT GROUP 3

ΛΘΔΞΠΣΦΨΩ∇∑∏∫ΠΣ
γδεζηθκλμνξπρστφχψωφ∂ϑϖυχϱϵι

GREEK LIGHT INCLINED GROUP 4

ΓΛΘΔΞΠΣΦΨΩΔ
αβγδεζηθκλμνξπρστφχψωφ

GREEK BOLD UPRIGHT GROUP 5

ΓΔΘΞΛΠΣΥΦΨΩ∇ςεςε
αβγδεζηθκλμνξπρστφχψωφϑϖυςϱι

GREEK BOLD INCLINED GROUP 6

ΓΔΘΞΛΠΣΥΦΨΩς
αβγδεζηθκλμνξπρσ υφϑϖυςϱι

GREEK LIGHT UPRIGHT GROUP 8

ΓΔΘΛΞΠΣΦΨΩ∇|αβγδεζηθκλμνξπρς φχψωφ

GREEK LIGHT INCLINED GROUP 9

ΓΔΘΛΞΠΣΦΨΩαβγδεζηθκλμνξπρστφχψωφ

LATIN LIGHT ITALIC GROUP 10

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz

LATIN LIGHT ROMAN GROUP 10

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 1234567890
()[]∪∩⊕⊖·⊗∅∧∨•√

LATIN LIGHT SCRIPT GROUP 18

𝒜ℬ𝒞𝒟ℰℱ𝒢ℋℐ𝒥𝒦ℒℳ𝒩𝒪𝒫𝒬
ℛ𝒮𝒯𝒰𝒱𝒲𝒳𝒴𝒵
𝒶𝒷𝒸𝒹ℯ𝒻ℊ𝒽𝒾𝒿𝓀𝓁𝓂𝓃ℴ𝓅𝓆𝓇𝓈𝓉𝓊𝓋𝓌𝓍𝓎𝓍

LATIN BOLD SCRIPT 20

𝓐𝓑𝓒𝓓𝓔𝓕𝓖𝓗𝓘𝓙𝓚𝓛𝓜𝓝𝓞𝓟𝓠
𝓡𝓢𝓣𝓤𝓥𝓦𝓧𝓨𝓩
𝓪𝓫𝓬𝓭𝓮𝓯𝓰𝓱𝓲𝓳𝓴𝓵𝓶𝓷𝓸𝓹𝓺𝓻𝓼𝓽𝓾𝓿𝔀𝔁𝔂𝔁

FRACTURE LIGHT GROUP 22

𝔄𝔅ℭ𝔇𝔈𝔉𝔊ℌℑ𝔍𝔎𝔏𝔐𝔑𝔒𝔓𝔔ℜ𝔖𝔗𝔘𝔙𝔚𝔛𝔜ℨ
𝔞𝔟𝔠𝔡𝔢𝔣𝔤𝔥𝔦𝔧𝔨𝔩𝔪𝔫𝔬𝔭𝔮𝔯𝔰𝔱𝔲𝔳𝔴𝔵𝔶𝔷

FRACTURE BOLD GROUP 23

𝕬𝕭𝕮𝕯𝕰𝕱𝕲𝕳𝕴𝕵𝕶𝕷𝕸𝕹𝕺𝕻𝕼𝕽𝕾𝕿𝖀𝖁𝖂𝖃𝖄𝖅
𝖆𝖇𝖈𝖉𝖊𝖋𝖌𝖍𝖎𝖏𝖐𝖑𝖒𝖓𝖔𝖕𝖖𝖗𝖘𝖙𝖚𝖛𝖜𝖝𝖞𝖟

LATIN OPEN FACE GROUP 35

𝔸𝔹ℂ𝔻𝔼𝔽𝔾ℍ𝕀𝕁𝕂𝕃𝕄ℕ𝕆ℙℚℝ𝕊𝕋𝕌𝕍𝕎𝕏𝕐ℤ
ghkst1(])

LARGE DISPLAY CHARACTERS

()[]{}⟨⟩[]/√∮‖

PHONETICS SYMBOLS

ɑɑɒɑƀɔɕχɖðɗ
ʤɖɛɜɘɟɡɓɦɧɥɦɨɫɬɮɱɯɰɲŋɳɵɵʊϙφ
qʁɹɾβʂʃtɵθʧɲɹɯʌʋɣʎʏzʒʓ˞ː ʻ ʼ¿¡